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Irrotational and Incompressible Binary Systems in the First Post-Newtonian Approximation of General Relativity

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The first post-Newtonian (PN) hydrostatic equations for an irrotational fluid are solved for an incompressible binary system. The equilibrium configuration of the binary system is given by a small deformation from the irrotational Darwin-Riemann ellipsoid which is the solution at Newtonian order. It is found that the orbital separation at the innermost stable circular orbit (ISCO) decreases when one increases the compactness parameter M_*/c^2a_*, in which M_* and a_* denote the mass and the radius of a star, respectively. If we compare the 1PN angular velocity of the binary system at the ISCO in units of $\sqrt{M_*/a_*^3}$ with that of Newtonian order, the angular velocity at the ISCO is almost the same value as that at Newtonian order when one increases the compactness parameter. Also, we do not find the instability point driven by the deformation at 1PN order, where a new sequence bifurcates throughout the equilibrium sequence of the binary system until the ISCO.

We also investigate the validity of an ellipsoidal approximation, in which a 1PN solution is obtained assuming an ellipsoidal figure and neglecting the deformation. It is found that the ellipsoidal approximation gives a fairly accurate result for the total energy, total angular momentum and angular velocity. However, if we neglect the velocity potential of 1PN order, we tend to overestimate the angular velocity at the ISCO regardless of the shape of the star (ellipsoidal figure or deformed figure).

I. INTRODUCTION

At the beginning of the next century, we will have powerful instruments for the detection of gravitational waves, such as LIGO, [1] VIRGO, [2] GEO [3] and TAMA. [4] One of the most promising sources of gravitational waves in the sensitive frequency range of these laser interferometers is coalescing binary neutron stars (BNSs). If we have accurate theoretical templates of inspiraling phase of BNSs, we can extract various information regarding them from gravitational waves, such as their mass and spin. [5] Moreover, when we construct reliable theoretical templates around the innermost stable circular orbit (ISCO) of BNSs, we can obtain physical information about the equation of state of neutron stars, i.e., the relation between the mass and the radius of a neutron star. [6]

Binary neutron stars evolve due to the radiation reaction of gravitational waves, so that they cannot reach equilibrium states. However, the timescale of the orbital decay is much longer than the orbital period of BNSs until the ISCO. Therefore, we can regard the state of a binary system as a quasi-equilibrium, even if the orbit approaches the ISCO.

From this point of view, several authors have attempted to obtain relativistic quasi-equilibrium configurations of BNSs numerically. [7,8] However, they assume that the binary system is synchronized. Synchronization (or corotation) is not a realistic assumption for the BNSs velocity field just outside the ISCO. This is because the viscosity of a neutron star is negligible even near the ISCO, and the velocity field of BNSs becomes irrotational or nearly irrotational. [9,10]

Using the Newtonian theory, Uryū and Eriguchi have recently investigated an irrotational binary system. [11–13] However, since BNSs are general relativistic objects, the Newtonian treatment is not sufficient and we need to include general relativistic effects. In order to investigate general relativistic effects on the orbital motion of BNSs, equilibrium sequences of BNSs composed of Newtonian stars with interaction forces of 2PN order have been studied. [14] On the other hand, Wilson, Mathews and Marronetti have computed sequences of irrotational BNSs in the conformally flat approximation of general relativity. [15] They suggest that the central densities of the stars increase when the stars decrease their separation and massive neutron stars collapse to black holes prior to merger. This conclusion is in

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contradicting with that of other works. [16,8,17] Therefore, it is necessary to solve the quasi-equilibrium problem in general relativity for an irrotational binary system.

In order to obtain accurate theoretical templates of gravitational waves from irrotational BNSs just prior to the ISCO, which is our goal, a general relativistic hydrostatic problem with compressible equation of state must be numerically solved. Even if we could obtain such numerical solutions, it is necessary to compare them with analytic or semi-analytic ones in order to check the validity of the numerical calculation. Lombardi, Rasio and Shapiro have semi-analytically studied irrotational BNSs using the energy variational method in the first post-Newtonian (PN) approximation of general relativity. [17] However, they do not solve the velocity fields of the binary system at 1PN order. They only give the velocity fields of the incompressible fluid at Newtonian order which flows along a plane perpendicular to the rotational axis x_3. We know that the existence of the x_3 component of the velocity field is important in the irrotational fluid problem. [11–13,18] From this fact, we believe the velocity potential of 1PN order in the irrotational binary problem is important.

We have already determined the equilibrium sequence of a corotating binary system. [19] However, the formalism which we have used in these papers is not applicable for the irrotational case. A formalism for obtaining quasi-equilibrium configuration of an irrotational binary system in general relativity has recently been constructed. [20–24] In this formalism, we need to solve only two hydrostatic equations as for the fluid equations. One of them consists of the integrated forms of the Euler equation and the other is the Poisson equation for the velocity potential. Thus, the formalism seems to be very tractable for computing the equilibrium configuration of realistic irrotational bodies. In order to develop the method for solving the irrotational binary problem consistently, the present author and collaborators recently investigated an irrotational and incompressible star of 1PN order, [18] using above mentioned formalism. Our method was originally introduced by Chandrasekhar. [25–28] As an extension of this method, we study an irrotational and incompressible binary system in the 1PN approximation of general relativity in this paper. We assume that each star in the binary system is constructed from an incompressible and homogeneous fluid. This assumption presents a great advantage for solving equations analytically. Also, the deformation of the figure of the binary system at 1PN order is given by the Lagrangian displacement vectors.

This paper is organized as follows. In §2, we formulate the method to solve the irrotational binary problem. In §3, the Lagrangian displacement vectors for description of the deformation of the binary system are given, and the angular velocity of 1PN order is derived from the first tensor virial relation. We give the boundary conditions for determining the velocity potential and the deformed figure of 1PN order in §4. The total energy and total angular momentum of the binary system are calculated in §5 and numerical results are given in §6. Section 7 is devoted to summary and discussion.

Throughout this paper, c denotes the light velocity and we use units in which $G = 1$. Latin indices i, j, k, $\cdots$ take values 1 to 3, and δ_{ij} denotes the Kronecker delta. We use I_{ij} and $\not{I}_{ij}$ as the quadrupole moment and its trace free part,

$$\not{I}_{ij} = I_{ij} - \frac{1}{3}\delta_{ij}\sum_{k=1}^{3} I_{kk} \tag{1.1}$$

of each star in the binary system.

II. FORMULATION

Equilibrium configurations of binary systems with non-uniform velocity fields are obtained by solving the Euler, continuity and Poisson equations consistently. Since we consider a binary system composed of incompressible stars, all the calculations are carried out analytically, even in the 1PN case. The procedure is as follows.

(1) We construct a Newtonian equilibrium configuration of the binary system, i.e., the irrotational Darwin-Riemann ellipsoid, [29–31] as a non-perturbed state. For simplicity, we consider only the case in which the directions of the vorticity vectors and the angular velocity vector lie along x_3-axis, and we assume that BNSs are composed of equal mass stars.

(2) The 1PN corrections for the velocity potential and gravitational potentials are obtained from their 1PN Poisson equations.

(3) We calculate the deformation from the Newtonian binary system induced by 1PN gravity using the Lagrangian displacement vectors introduced by Chandrasekhar. [29] Then, the corrections for the Newtonian quantities due to the deformation of the binary system are estimated.

(4) From the first tensor virial relation at 1PN order, i.e., the force balance equation at 1PN order, we calculate the correction for the angular velocity. [19]

(5) We substitute all the 1PN corrections obtained in (2), (3) and (4) into the 1PN Euler and continuity equations. Then, the coefficients of the Lagrangian displacement vectors and the 1PN velocity potential are determined from these two equations with the boundary conditions on the stellar surface.

In this section, we calculate the Newtonian and 1PN terms which we need in the above procedures. We consider the equilibrium sequences of BNSs of equal masses ($M_1 = M_2 = M$) whose coordinate separation is R.[†] For calculational simplicity, we use four kinds of coordinate systems which are all rotating frames. The first frame is x_i, in which the center of mass of a star (star 1) is located at the origin of the coordinate system, and the other one (star 2) is located at $(x_1,\ x_2,\ x_3) = (-R,\ 0,\ 0)$ (see Fig. 1). The second one is y_i, in which the center of mass of star 2 is located at the origin, and star 1 is located at $(y_1,\ y_2,\ y_3) = (R,\ 0,\ 0)$. The third one is X_i, in which the origin is located at the center of mass of the binary system, and the centers of masses of two stars are located at $(X_1,\ X_2,\ X_3) = (R/2,\ 0,\ 0)$ and $(-R/2,\ 0,\ 0)$. The fourth one is Y_i. This frame is just the same rotating frame as X_i, but we label it with a different name for convenience. We use X_i for star 1 and Y_i for star 2. Then, the relations among the four corotating frames are

$$\left(X_1,\ X_2,\ X_3\right) = \left(x_1 + \frac{R}{2},\ x_2,\ x_3\right), \tag{2.1}$$

$$\left(Y_1,\ Y_2,\ Y_3\right) = \left(y_1 - \frac{R}{2},\ y_2,\ y_3\right). \tag{2.2}$$

Due to the symmetry, we consider only the equilibrium configuration of star 1 in the following.

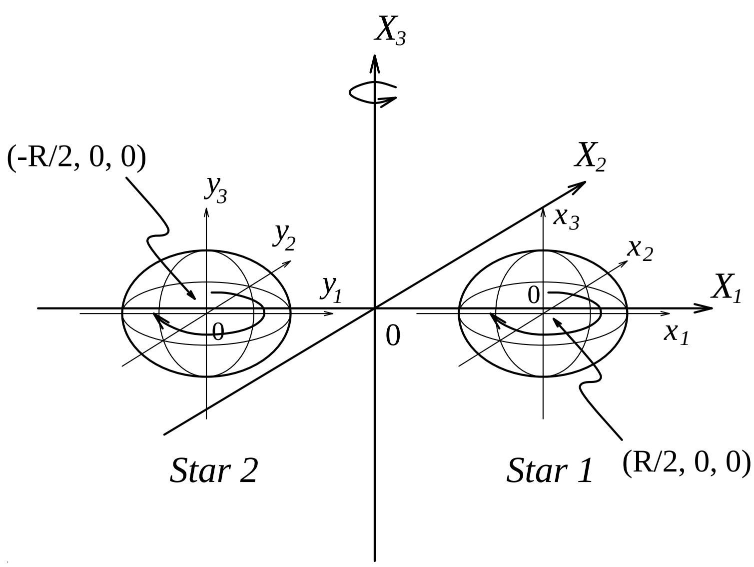

FIG. 1. Sketch of the Darwin-Riemann ellipsoid. Three coordinate systems are shown. The center of mass of star 1 is located at the origin of the corotating frame x_i, and that of star 2 is located at the origin of y_i. These two origins of the corotating frames are located at $(R/2, 0, 0)$ and $(-R/2, 0, 0)$ in the coordinate system X_i in which the origin is located at the center of mass of the binary system.

The main purpose of this paper is to calculate the 1PN corrections for the angular velocity, the velocity potential, the energy, and the angular momentum in the Darwin-Riemann problem. In the calculation, we assume that the parameter a_0/R, where a_0 is a typical radius of a star, is small and we use it as the expansion parameter. Accordingly, there exist two different types of expansion parameters in this paper. One of them is the post-Newtonian parameter and the other is a_0/R.

At Newtonian order, the angular velocity ($\Omega_{\rm DR}$) becomes [32] (see §III for derivation)

$$\Omega_{\rm DR}^2 = \frac{2M}{R^3} + \frac{18 \not{I}_{11}}{R^5} + O(R^{-7}). \tag{2.3}$$

[†]The coordinate condition in this paper is the standard PN one. [25]

Thus, in the 1PN approximation, we can expect that the following types of quantities will be the main terms at 1PN order:

$$\sim \frac{M}{R^3} \times \frac{M}{a_0 c^2}, \quad \sim \frac{M}{R^3} \times \frac{M}{Rc^2}, \quad \sim \frac{M a_0^2}{R^5} \times \frac{M}{a_0 c^2}, \text{ and } \sim \frac{M a_0^2}{R^5} \times \frac{M}{Rc^2}. \tag{2.4}$$

Here we have used the relation $\not{I}_{11} \sim M a_0^2$. We will derive the four types of terms given above.

Since we consider an incompressible fluid, the gravitational potentials inside each star are expressed as polynomials in the coordinates x_i. For the purpose of obtaining the 1PN corrections given above, we need to take into account the coefficients of terms such as $x_1^{m_1} x_2^{m_2} x_3^{m_3}$ in the gravitational potentials of 1PN order, where $0 \le m_1,\ m_2,\ m_3 \le 5$ and $0 \le m_1 + m_2 + m_3\ (\equiv m_t) \le 5$ up to $O(R^{-5})$ for the case in which m_t is odd and up to $O(R^{-3})$ for the case in which m_t is even.

A. Hydrostatic and Poisson equations for 1PN irrotational binary systems

For an irrotational fluid, the relativistic Euler equation can be integrated, and in the 1PN case, it is written as [22]

$$\begin{aligned}
\text{constant} = {} & \frac{P}{\rho} - U + \frac{1}{2}\sum_k (\partial_k \phi_{\rm N})^2 - \sum_k \ell^k \partial_k \phi_{\rm N} \\
& + \frac{1}{c^2}\Big[-\frac{P}{\rho} U + \frac{1}{2} U^2 + X - \frac{1}{2}\Big(\frac{P}{\rho} + 3U\Big) \sum_k (\partial_k \phi_{\rm N})^2 - \frac{1}{8}\Big(\sum_k (\partial_k \phi_{\rm N})^2\Big)^2 \\
& + \sum_k (\partial_k \phi_{\rm N})(\partial_k \phi_{\rm PN}) - \sum_k \ell^k \partial_k \phi_{\rm PN} - \sum_k \hat{\beta}_k \partial_k \phi_{\rm N} \Big],
\end{aligned} \tag{2.5}$$

where we have assumed that the fluid is incompressible, i.e., $\rho =$ const. In Eq. (2.5), P, ρ, ℓ^k, $\phi_{\rm N}$, $\phi_{\rm PN}$, U, X, and $\hat{\beta}_k$ denote the pressure, the density, the velocity field of the orbital motion (spatial component of the timelike Killing vector), the Newtonian and 1PN velocity potentials, and the last three terms are the Newtonian and 1PN potentials, which are derived by solving the Poisson equations as

$$\Delta U = -4\pi\rho, \tag{2.6}$$

$$\Delta X = 4\pi\rho\Big[2U + \frac{3P}{\rho} + 2\sum_k (\partial_k \phi_{\rm N})^2\Big], \tag{2.7}$$

$$\Delta P_k = -4\pi\rho \partial_k \phi_{\rm N}, \tag{2.8}$$

$$\Delta \chi = 4\pi\rho \sum_k (\partial_k \phi_{\rm N}) X^k. \tag{2.9}$$

Here $\hat{\beta}_k$ is expressed as

$$\hat{\beta}_k = -\frac{7}{2} P_k + \frac{1}{2}\Big(\partial_k \chi + \sum_l X^l \partial_k P_l\Big). \tag{2.10}$$

We note that using the gravitational potentials, the spacetime line element to 1PN order can be written as

$$ds^2 = -\alpha^2 c^2 dt^2 + \frac{2}{c^2}\sum_i \hat{\beta}_i dx^i dt + \Big(1 + \frac{2U}{c^2}\Big)\sum_i dx^i dx^i, \tag{2.11}$$

where

$$\alpha = 1 - \frac{U}{c^2} + \frac{1}{c^4}\Big(\frac{U^2}{2} + X\Big) + O(c^{-6}). \tag{2.12}$$

A characteristic feature of the irrotational fluid is that the continuity equation reduces to a Poisson-type equation for a velocity potential ϕ, [22,23] and in the 1PN incompressible case, it becomes

$$\sum_i \rho \partial_i C_i = 0, \tag{2.13}$$

where

$$C_i = -\ell^i + \partial_i \phi_{\rm N} + \frac{1}{c^2}\Big[-\ell^i\Big(\frac{1}{2}\sum_k (\partial_k\phi_{\rm N})^2 + 3U\Big) - \frac{P}{\rho}\partial_i\phi_{\rm N} - \hat{\beta}_i + \partial_i\phi_{\rm PN}\Big]. \tag{2.14}$$

In the following subsections, we obtain the terms appearing in Eq. (2.5) by solving equations separately.

B. Newtonian terms

Each gravitational potential is composed of two parts. One of them is the contribution from star 1 and the other is from star 2. In the following, we denote the former part as $\Psi^{1\to1}$ and the latter one as $\Psi^{2\to1}$, where Ψ denotes one of the potentials. We also define $\Psi^{1\to2}$ and $\Psi^{2\to2}$ as the contribution from star 1 to 2 and star 2 to itself, respectively.

In analogy to previous studies, [29–31] the configuration of each star in the binary system at Newtonian order is assumed to be an ellipsoidal figure of its axial length a_1, a_2, and a_3. In this case, the solution of the Poisson equation for the Newtonian gravitational potential

$$\Delta U = -4\pi\rho, \tag{2.15}$$

is written as $U = U^{1\to1} + U^{2\to1}$, where

$$U^{1\to1} = \pi\rho(A_0 - \sum_l A_l x_l^2), \tag{2.16}$$

$$\begin{aligned} U^{2\to1} = \frac{M}{R}\Big[&1 - \frac{x_1}{R} + \frac{2x_1^2 - x_2^2 - x_3^2}{2R^2} + \frac{-2x_1^3 + 3x_1(x_2^2+x_3^2)}{2R^3} \\ &+\frac{8x_1^4 + 3x_2^4 + 3x_3^4 - 24x_1^2(x_2^2+x_3^2) + 6x_2^2x_3^2}{8R^4}\Big] \\ &+\frac{3{I\!\!\!-}_{11}}{2R^3}\Big(1 - \frac{3x_1}{R} + \frac{12x_1^2 - 5x_2^2 - 5x_3^2}{2R^2}\Big) + \frac{3}{2R^5}({I\!\!\!-}_{22}x_2^2 + {I\!\!\!-}_{33}x_3^2) \\ &+\frac{1}{8R^5}(8I_{1111} + 3I_{2222} + 3I_{3333} - 24I_{1122} - 24I_{1133} + 6I_{2233}) \\ &+O(R^{-6}), \end{aligned} \tag{2.17}$$

and

$$I_{iijj} = \int d^3x\rho x_i^2 x_j^2 = \frac{M}{35}a_i^2 a_j^2(1+2\delta_{ij}). \tag{2.18}$$

Here, $A_{ij\cdots}$ are index symbols introduced by Chandrasekhar [29] and $A_0 = \sum_l A_l a_l^2$ is calculated from [29]

$$\begin{aligned} A_0 &= a_1a_2a_3\int_0^\infty \frac{du}{\sqrt{(a_1^2+u)(a_2^2+u)(a_3^2+u)}} \\ &= a_1^2\alpha_2\alpha_3\int_0^\infty \frac{dt}{\sqrt{(1+t)(\alpha_2^2+t)(\alpha_3^2+t)}}, \end{aligned} \tag{2.19}$$

where $\alpha_2 = a_2/a_1$ and $\alpha_3 = a_3/a_1$. Note that $U^{2\to2}$ and $U^{1\to2}$ are obtained by changing x_1 to $-(x_1+R)$ in $U^{1\to1}$ and $U^{2\to1}$, respectively.

The pressure at Newtonian order is written as

$$P = P_0\Big(1 - \sum_l \frac{x_l^2}{a_l^2}\Big), \tag{2.20}$$

where P_0 denotes the pressure at the center at the star and is calculated from the scalar virial relation as

$$\frac{P_0}{\rho} = \frac{1}{3}\pi\rho A_0 - \frac{5}{2R^3}{I\!\!\!-}_{11} - \frac{1}{6}F_a(a_1^2 - a_2^2)\Omega_{\rm DR}^2 + O(R^{-5}). \tag{2.21}$$

The axial ratios α_2 and α_3 are determined from [31]

$$\begin{aligned}
a_1^2 A_1 &= \frac{P_0}{\pi\rho^2} + \frac{M a_1^2}{\pi\rho R^3} + \frac{a_1^2}{2\pi\rho}\Omega_{\rm DR}^2 F_a(2-F_a) + O(R^{-5}),\\
a_2^2 A_2 &= \frac{P_0}{\pi\rho^2} - \frac{M a_2^2}{2\pi\rho R^3} - \frac{a_2^2}{2\pi\rho}\Omega_{\rm DR}^2 F_a(2+F_a) + O(R^{-5}),\\
a_3^2 A_3 &= \frac{P_0}{\pi\rho^2} - \frac{M a_3^2}{2\pi\rho R^3} + O(R^{-5}).
\end{aligned} \tag{2.22}$$

We can determine the equilibrium sequence of the irrotational Darwin-Riemann ellipsoid from Eqs. (2.3) and (2.22).

The velocity field of Newtonian order of star 1 in the inertial frame is written as $v_i \equiv \partial_i \phi_{\rm N}$, where

$$\begin{aligned}
v_1 &= \ell_1 + u_1 = -\Big(\frac{a_1^2 f_R}{a_1^2 + a_2^2} + 1\Big)\Omega x_2,\\
v_2 &= \ell_2 + u_2 = \Big(\frac{a_2^2 f_R}{a_1^2 + a_2^2} + 1\Big)\Omega x_1 + \frac{R}{2}\Omega,\\
v_3 &= 0.
\end{aligned} \tag{2.23}$$

The quantity u_i is the internal velocity of the star in the corotating frame x_i, and Ω denotes the angular velocity of the binary system. ℓ_i and u_i are given by

$$\ell_i = \Big(-\Omega x_2,\ \Omega\Big(x_1 + \frac{R}{2}\Big),\ 0\Big), \tag{2.24}$$

$$\begin{aligned}
u_i &= \Big(\frac{a_1}{a_2}\Lambda x_2,\ \ -\frac{a_2}{a_1}\Lambda x_1,\ 0\Big)\\
&= \Big(-\frac{a_1^2 f_R}{a_1^2 + a_2^2}\Omega x_2,\ \frac{a_2^2 f_R}{a_1^2 + a_2^2}\Omega x_1,\ 0\Big),
\end{aligned} \tag{2.25}$$

where Λ is the angular velocity of the internal motion. f_R is defined as ζ/Ω, where $\zeta \equiv (\mathrm{rot}\mathbf{u})_3$ denotes the vorticity in the corotating frame. For the irrotational case, f_R becomes -2. Thus, the velocity potential at Newtonian order becomes

$$\phi_{\rm N} = F_a \Omega x_1 x_2 + \frac{R}{2}\Omega x_2 \tag{2.26}$$

and the velocity field is written as

$$v_i = \Big(F_a \Omega x_2,\ F_a \Omega x_1 + \frac{R}{2}\Omega,\ 0\Big), \tag{2.27}$$

where we define

$$F_a \equiv \frac{a_1^2 - a_2^2}{a_1^2 + a_2^2}. \tag{2.28}$$

C. 1PN terms

1. X

For computational convenience, we separate X into two parts as

$$X = X_0 + X_v, \tag{2.29}$$

where X_0 and X_v are derived from the Poisson-like equations

$$\Delta X_0 = 4\pi\rho\Big(2U + \frac{3P}{\rho}\Big), \tag{2.30}$$

$$\Delta X_v = 8\pi\rho \sum_k (\partial_k \phi_k)^2. \tag{2.31}$$

As in the case of U, the 1PN potentials are divided into two parts as $X_0 = X_0^{1\to1} + X_0^{2\to1}$, and we consider them separately.

The contribution from star 1 is derived from the Poisson-like equation

$$\Delta X_0^{1\to1} = 4\pi\rho\Big[\Big(2\pi\rho A_0 + \frac{3P_0}{\rho}\Big) - \sum_l \Big(2\pi\rho A_l + \frac{3P_0}{\rho a_l^2}\Big) x_l^2 + 2U^{2\to1}\Big], \tag{2.32}$$

and the solution is

$$\begin{aligned} X_0^{1\to1} &= -\alpha_0 U^{1\to1} + \alpha_1 D_1 + \sum_l \eta_l D_{ll} - \frac{M}{R^3}(2D_{11} - D_{22} - D_{33}) \\ &\quad + \frac{M}{R^4}(2D_{111} - 3D_{122} - 3D_{133}) + O(R^{-5}), \end{aligned} \tag{2.33}$$

where

$$\alpha_0 = 2\pi\rho A_0 + \frac{3P_0}{\rho} + \frac{2M}{R} + \frac{3\not{I}_{11}}{R^3} + O(R^{-5}), \tag{2.34}$$

$$\alpha_1 = \frac{2M}{R^2} + \frac{9\not{I}_{11}}{R^4} + O(R^{-6}), \tag{2.35}$$

$$\eta_l = 2\pi\rho A_l + \frac{3P_0}{\rho a_l^2}. \tag{2.36}$$

Here, $D_{ij\cdots}$ are the solutions of equations

$$\Delta D_{ij\cdots} = -4\pi\rho x_i x_j \cdots, \tag{2.37}$$

and the solutions are written as [29]

$$D_i = \pi\rho a_i^2 \big(A_i - \sum_l A_{il} x_l^2\big) x_i, \tag{2.38}$$

$$\begin{aligned} D_{ii} &= \pi\rho\Big[a_i^4\big(A_{ii} - \sum_l A_{iil} x_l^2\big) x_i^2 \\ &\quad + \frac{1}{4} a_i^2 \big(B_i - 2\sum_l B_{il} x_l^2 + \sum_l \sum_m B_{ilm} x_l^2 x_m^2\big)\Big], \end{aligned} \tag{2.39}$$

$$D_{ij} = \pi\rho a_i^2 a_j^2 \big(A_{ij} - \sum_l A_{ijl} x_l^2\big) x_i x_j, \quad (i \neq j) \tag{2.40}$$

$$\begin{aligned} D_{iii} &= \pi\rho\Big[a_i^6\big(A_{iii} - \sum_l A_{iiil} x_l^2\big) x_i^3 \\ &\quad + \frac{3}{4} a_i^4 \big(B_{ii} - 2\sum_l B_{iil} x_l^2 + \sum_l \sum_m B_{iilm} x_l^2 x_m^2\big) x_i\Big], \end{aligned} \tag{2.41}$$

$$\begin{aligned} D_{ijj} &= \pi\rho\Big[a_i^2 a_j^4\big(A_{ijj} - \sum_l A_{ijjl} x_l^2\big) x_i x_j^2 \\ &\quad + \frac{1}{4} a_i^2 a_j^2 \big(B_{ij} - 2\sum_l B_{ijl} x_l^2 + \sum_l \sum_m B_{ijlm} x_l^2 x_m^2\big) x_i\Big], \quad (i \neq j) \end{aligned} \tag{2.42}$$

where $B_{ijk\cdots}$ are index symbols defined by Chandrasekhar. [29]

The contribution from star 2 for X_0 is calculated from the equation

$$\Delta X_0^{2\to1} = 4\pi\rho\Big[\Big(2\pi\rho A_0 + \frac{3P_0}{\rho}\Big) - \sum_l \Big(2\pi\rho A_l + \frac{3P_0}{\rho a_l^2}\Big) y_l^2 + 2U^{1\to2}\Big], \tag{2.43}$$

where $y_1 = -(x_1 + R)$, $y_2 = x_2$, and $y_3 = x_3$. The solution is written

$$\begin{aligned} X_0^{2\to1} &= -\alpha_0 U^{2\to1} - \alpha_1 D_1^{2\to1} + \sum_l \eta_l D_{ll}^{2\to1} \\ &\quad - \frac{M}{R^3}(2D_{11}^{2\to1} - D_{22}^{2\to1} - D_{33}^{2\to1}) + O(R^{-6}), \end{aligned} \tag{2.44}$$

where $D^{2\to 1}_{ij\cdots}$ are calculated from the same equations as in the case of $D_{ij\cdots}$, i.e.,

$$\Delta D^{2\to 1}_{ij\cdots} = -4\pi\rho y_i y_j \cdots . \tag{2.45}$$

The solutions are

$$\begin{aligned} D^{2\to 1}_1 &= \frac{I_{11}}{R^2}\Big[1 - \frac{2x_1}{R} + \frac{3}{2R^2}(2x_1^2 - x_2^2 - x_3^2) - \frac{2}{R^3}(2x_1^3 - 3x_1x_2^2 - 3x_1x_3^2)\Big] \\ &\quad + \frac{1}{2R^4}(2I_{1111} - 3I_{1122} - 3I_{1133})\Big(1 - \frac{4x_1}{R}\Big) + O(R^{-6}), \end{aligned} \tag{2.46}$$

$$D^{2\to 1}_i = \frac{I_{ii}}{R^3}x_i\Big(1 - \frac{3x_1}{R}\Big) + O(R^{-5}), \quad (i \neq 1) \tag{2.47}$$

$$\begin{aligned} D^{2\to 1}_{ii} &= \frac{I_{ii}}{R}\Big[1 - \frac{x_1}{R} + \frac{2x_1^2 - x_2^2 - x_3^2}{2R^2} + \frac{-2x_1^3 + 3x_1(x_2^2 + x_3^2)}{2R^3}\Big] \\ &\quad + \frac{3\not{I}_{ii11}}{2R^3}\Big(1 - \frac{3x_1}{R}\Big) + O(R^{-5}), \end{aligned} \tag{2.48}$$

$$D^{2\to 1}_{1i} = 3I_{11ii}\frac{x_i}{R^4}\Big(1 - \frac{4x_1}{R}\Big) + O(R^{-6}), \quad (i \neq 1) \tag{2.49}$$

$$D^{2\to 1}_{1jj} = \frac{I_{11jj}}{R^2}\Big[1 - \frac{2x_1}{R} + \frac{3}{2R^2}(2x_1^2 - x_2^2 - x_3^2)\Big] + O(R^{-5}), \tag{2.50}$$

$$D^{2\to 1}_{2jj} = \frac{I_{22jj}}{R^2}\Big[\frac{x_2}{R} - \frac{3x_1x_2}{R^2}\Big] + O(R^{-5}), \tag{2.51}$$

$$D^{2\to 1}_{3jj} = \frac{I_{33jj}}{R^2}\Big[\frac{x_3}{R} - \frac{3x_1x_3}{R^2}\Big] + O(R^{-5}), \tag{2.52}$$

where

$$\not{I}_{ii11} = I_{ii11} - \frac{1}{3}\sum_l I_{iill}. \tag{2.53}$$

As in the case of X_0, it is convenient to separate X_v as $X^{1\to 1}_v + X^{2\to 1}_v$. Here, $X^{1\to 1}_v$ is derived from the Poisson-like equation

$$\Delta X^{1\to 1}_v = 8\pi\rho\Omega^2\Big[F_a^2(x_1^2 + x_2^2) + F_a R x_1 + \frac{R^2}{4}\Big], \tag{2.54}$$

and the solution is written

$$X^{1\to 1}_v = -2\Omega^2\Big[F_a^2(D_{11} + D_{22}) + F_a R D_1 + \frac{R^2}{4}U^{1\to 1}\Big]. \tag{2.55}$$

The equation for $X^{2\to 1}_v$ is

$$\Delta X^{2\to 1}_v = 8\pi\rho\Omega^2\Big[F_a^2(y_1^2 + y_2^2) - F_a R y_1 + \frac{R^2}{4}\Big]. \tag{2.56}$$

Then, the solution is found to be

$$X^{2\to 1}_v = -2\Omega^2\Big[F_a^2(D^{2\to 1}_{11} + D^{2\to 1}_{22}) - F_a R D^{2\to 1}_1 + \frac{R^2}{4}U^{2\to 1}\Big]. \tag{2.57}$$

2. $\hat\beta_k$

Substituting v_i of Newtonian order into the equations for P_i and χ, we immediately find the solutions

$$
\begin{aligned}
P_1^{1\to1} &= F_a \Omega D_2, && (2.58)\\
P_2^{1\to1} &= \Omega\Big(F_a D_1 + \frac{R}{2} U^{1\to1}\Big), && (2.59)\\
P_3^{1\to1} &= 0, && (2.60)\\
P_1^{2\to1} &= F_a \Omega D_2^{2\to1}, && (2.61)\\
P_2^{2\to1} &= \Omega\Big(F_a D_1^{2\to1} - \frac{R}{2} U^{2\to1}\Big), && (2.62)\\
P_3^{2\to1} &= 0, && (2.63)\\
\chi^{1\to1} &= -\Omega\Big[2F_a D_{12} + \frac{R}{2}(F_a+1)D_2\Big], && (2.64)\\
\chi^{2\to1} &= -\Omega\Big[2F_a D_{12}^{2\to1} - \frac{R}{2}(F_a+1)D_2^{2\to1}\Big]. && (2.65)
\end{aligned}
$$

Using the solutions of P_k and χ we have derived above, we obtain

$$
\hat\beta_k \equiv \hat\beta_k^{1\to1} + \hat\beta_k^{2\to1}, \tag{2.66}
$$

where

$$
\begin{aligned}
\hat\beta_1^{1\to1} = {}& \frac{F_a}{2}\pi\rho\Omega x_2\Big[a_1^2 A_1 - 7a_2^2 A_2 - 2a_1^2 a_2^2 A_{12}\\
&\quad +(5a_2^2 A_{12} - 3a_1^2 A_{11} + 6a_1^2 a_2^2 A_{112})x_1^2\\
&\quad +(7a_2^2 A_{22} - a_1^2 A_{12} + 2a_1^2 a_2^2 A_{122})x_2^2\\
&\quad +(7a_2^2 A_{23} - a_1^2 A_{13} + 2a_1^2 a_2^2 A_{123})x_3^2\Big]\\
& -\frac{R}{2}\pi\rho\Omega(A_1 - a_2^2 A_{12})x_1 x_2, && (2.67)\\
\hat\beta_2^{1\to1} = {}& \frac{F_a}{2}\pi\rho\Omega x_1\Big[a_2^2 A_2 - 7a_1^2 A_1 - 2a_1^2 a_2^2 A_{12}\\
&\quad +(7a_1^2 A_{11} - a_2^2 A_{12} + 2a_1^2 a_2^2 A_{112})x_1^2\\
&\quad +(5a_1^2 A_{12} - 3a_2^2 A_{22} + 6a_1^2 a_2^2 A_{122})x_2^2\\
&\quad +(7a_1^2 A_{13} - a_2^2 A_{23} + 2a_1^2 a_2^2 A_{123})x_3^2\Big]\\
& +\frac{R}{4}\pi\rho\Omega\Big[-7A_0 - a_2^2 A_2 + (7A_1 + a_2^2 A_{12})x_1^2 + (5A_2 + 3a_2^2 A_{22})x_2^2\\
&\quad +(7A_3 + a_2^2 A_{23})x_3^2\Big], && (2.68)\\
\hat\beta_3^{1\to1} = {}& -\frac{1}{2}\pi\rho\Omega x_2 x_3\Big[2F_a(a_1^2 A_{13} + a_2^2 A_{23} - 2a_1^2 a_2^2 A_{123})x_1\\
&\quad +R(A_3 - a_2^2 A_{23})\Big], && (2.69)\\
\hat\beta_1^{2\to1} = {}& \frac{\Omega}{2}\Big[\frac{Mx_2}{2R} - \frac{x_2}{R^3}\Big(2F_a I_{11} + 10F_a I_{22} + \frac{3}{2}I_{22} - \frac{9}{4}\not{I}_{11}\Big) - \frac{M}{R^2}x_1 x_2\\
&\quad +\frac{3M}{4R^3}x_2(2x_1^2 - x_2^2 - x_3^2) + O(R^{-4})\Big], && (2.70)\\
\hat\beta_2^{2\to1} = {}& \frac{\Omega}{2}\Big[\frac{7M}{2} + \frac{1}{R^2}\Big(-7F_a I_{11} + F_a I_{22} + \frac{I_{22}}{2} + \frac{21\not{I}_{11}}{4}\Big)\\
&\quad +\frac{x_1}{R}\Big\{-\frac{7M}{2} + \frac{1}{R^2}\Big(14F_a I_{11} - I_{22}\Big(2F_a + \frac{3}{2}\Big) - \frac{63\not{I}_{11}}{4}\Big)\Big\}\\
&\quad +\frac{7M}{4R^2}(2x_1^2 - x_2^2 - x_3^2) + \frac{M}{2R^2}x_2^2 - \frac{7M}{4R^3}x_1(2x_1^2 - 3x_2^2 - 3x_3^2)\\
&\quad -\frac{3M}{2R^3}x_1 x_2^2 + O(R^{-4})\Big], && (2.71)\\
\hat\beta_3^{2\to1} = {}& \frac{M\Omega}{4R^2}x_2 x_3\Big[1 - \frac{3x_1}{R} + O(R^{-2})\Big]. && (2.72)
\end{aligned}
$$

Then, the divergence of $\hat{\beta}_k$ is

$$\sum_k \partial_k \hat{\beta}_k = \sum_k \partial_k (\hat{\beta}_k^{1\to 1} + \hat{\beta}_k^{2\to 1}), \tag{2.73}$$

where

$$\sum_k \partial_k \hat{\beta}_k^{1\to 1} = 3\pi\rho\Omega\Big[2F_a(a_1^2+a_2^2)A_{12}x_1x_2 + RA_2x_2\Big], \tag{2.74}$$

$$\sum_k \partial_k \hat{\beta}_k^{2\to 1} = -\frac{3M}{2R^2}\Omega x_2\Big(1-\frac{3x_1}{R}+O(R^{-2})\Big). \tag{2.75}$$

Here we have used some relations among index symbols. [29]

3. $\phi_{\rm PN}$

An explicit form of the Poisson-like equation for $\phi_{\rm PN}$ can be written as

$$\begin{aligned}\Delta\phi_{\rm PN} &= \partial_i\Big[\ell^i\Big(\frac{1}{2}\sum_k(\partial_k\phi_{\rm N})^2+3U\Big)+\frac{P}{\rho}\partial_i\phi_{\rm N}+\hat{\beta}_i\Big],\\ &= -\frac{2P_0}{\rho}\frac{a_1^2-a_2^2}{a_1^2a_2^2}\Omega x_1x_2 - R\Omega\Big[\frac{a_2^2(a_1^2-a_2^2)}{(a_1^2+a_2^2)^2}\Omega^2+\frac{P_0}{\rho a_2^2}\Big]x_2\\ &\quad +\Omega\times O(R^{-4}).\end{aligned} \tag{2.76}$$

The solution of this equation can be written up to biquadratic terms in x_i as‡

$$\phi_{\rm PN} = (p+qx_1^2+rx_2^2+sx_3^2)x_1x_2 + (e+fx_1^2+gx_2^2+hx_3^2)x_2 + \text{const}, \tag{2.77}$$

where q, r, s, f, g, and h satisfy the conditions

$$3q+3r+s = -\frac{P_0}{\rho}\frac{a_1^2-a_2^2}{a_1^2a_2^2}\Omega, \tag{2.78}$$

$$f+3g+h = -\frac{R}{2}\Omega\Big[\frac{a_2^2(a_1^2-a_2^2)}{(a_1^2+a_2^2)^2}\Omega^2+\frac{P_0}{\rho a_2^2}\Big]. \tag{2.79}$$

On the right-hand side of Eq. (2.77), coefficients p, q, r, and s are concerned with the star itself. We can find these coefficients in a previous paper [18] in which we investigated irrotational and incompressible stars. On the other hand, the coefficients e, f, g, and h are associated with the binary motion.

D. Collection

Substituting the terms derived in the previous subsections into Eq. (2.5), we obtain

$$\begin{aligned}\mathcal{G} &\equiv \frac{P}{\rho} - U - \delta U\\ &\quad -\frac{1}{2}\Big(\Omega_{\rm DR}^2+\frac{1}{c^2}\delta\Omega^2\Big)\Big[F_a(2-F_a)x_1^2 - F_a(2+F_a)x_2^2 + Rx_1 + \frac{R^2}{4}\Big]\\ &\quad +\frac{1}{c^2}\Big[\gamma_0+\sum_l\gamma_l x_l^2+\sum_{l\le m}\gamma_{lm}x_l^2x_m^2 + x_1\big(\kappa_0+\sum_l\kappa_l x_l^2+\sum_{l\le m}\kappa_{lm}x_l^2x_m^2\big)\Big]\\ &= \text{const},\end{aligned} \tag{2.80}$$

‡Although we can add higher order terms which satisfy $\Delta\phi_{\rm PN}=0$, we neglect them for simplicity because we consider only biquadratic deformation of ellipsoids in this paper.

where $\Omega_{\rm DR}$ denotes the angular velocity of an irrotational Darwin-Riemann ellipsoid, and $\delta\Omega$ denotes the 1PN correction for the angular velocity. γ_{ij} and κ_{ij} are expressed as

$$
\begin{aligned}
\gamma_0 = {}& \pi\rho\Big[\frac{1}{2}\pi\rho A_0^2-\frac{P_0}{\rho}A_0-\Big(\frac{\alpha_0}{\pi\rho}+\frac{P_0}{\pi\rho^2}-A_0\Big)\Big(\frac{M}{R}+\frac{3\not{I}_{11}}{2R^3}\Big)-\alpha_0A_0 \\
&\quad +\frac{1}{4}\sum_l\eta_l a_l^2B_l+\frac{1}{\pi\rho}\sum_l\eta_l\Big(\frac{I_{ll}}{R}+\frac{3\not{I}_{ll11}}{2R^3}\Big) \\
&\quad -\frac{M}{4R^3}(2a_1^2B_1-a_2^2B_2-a_3^2B_3)-\frac{1}{2}\Omega^2F_a^2(a_1^2B_1+a_2^2B_2) \\
&\quad -\frac{1}{8}\Omega^2R^2\Big(\frac{P_0}{\pi\rho^2}-a_2^2A_2\Big)+O(R^{-5})\Big] \\
& +\frac{M}{2R^2}\Big(M-\frac{3\not{I}_{11}}{R^2}\Big)-\alpha_1\frac{I_{11}}{R^2}-\frac{R^4}{128}\Omega^4 \\
& -\frac{\Omega^2}{8}\Big[14MR+\frac{1}{R}\Big\{16F_a^2(I_{11}+I_{22})-30F_aI_{11}+(2F_a+1)I_{22}+21\not{I}_{11}\Big\}\Big] \\
& +O(R^{-6}),
\end{aligned}
\tag{2.81}
$$

$$
\begin{aligned}
\gamma_1 = {}& \pi\rho\Big[(\alpha_0-\pi\rho A_0)A_1+\frac{P_0}{\rho a_1^2}(A_0+a_1^2A_1)-\frac{1}{a_1^2}\Big(a_1^2A_1-\frac{P_0}{\pi\rho^2}\Big)\Big(\frac{M}{R}+\frac{3\not{I}_{11}}{2R^3}\Big) \\
&\quad -\Big(\frac{\alpha_0}{\pi\rho}+\frac{P_0}{\pi\rho^2}-A_0\Big)\frac{M}{R^3}+\eta_1a_1^2\Big(a_1^2A_{11}-\frac{1}{2}B_{11}\Big)-\frac{1}{2}\eta_2a_2^2B_{12} \\
&\quad -\frac{1}{2}\eta_3a_3^2B_{13}+\frac{1}{\pi\rho R^3}\sum_l\eta_lI_{ll} \\
&\quad -\frac{M}{2R^3}(4a_1^4A_{11}-2a_1^2B_{11}+a_2^2B_{12}+a_3^2B_{13}) \\
&\quad -\frac{1}{2}\Omega^2\Big\{\frac{R^2}{4}\Big(a_2^2A_{12}-\frac{P_0}{\pi\rho^2a_1^2}\Big)+F_a^2\Big(3A_0-7a_1^2A_1+a_2^2A_2 \\
&\qquad\qquad +4a_1^4A_{11}-2a_1^2a_2^2A_{12}-2a_1^2B_{11}-2a_2^2B_{12}+\frac{P_0}{\pi\rho^2}\Big)\Big\}\Big] \\
& +(F_a-1)\Omega p+O(R^{-4}),
\end{aligned}
\tag{2.82}
$$

$$
\begin{aligned}
\gamma_2 = {}& \pi\rho\Big[(\alpha_0-\pi\rho A_0)A_2+\frac{P_0}{\rho a_2^2}(A_0+a_2^2A_2)-\frac{1}{a_2^2}\Big(a_2^2A_2-\frac{P_0}{\pi\rho^2}\Big)\Big(\frac{M}{R}+\frac{3\not{I}_{11}}{2R^3}\Big) \\
&\quad +\Big(\frac{\alpha_0}{\pi\rho}+\frac{P_0}{\pi\rho^2}-A_0\Big)\frac{M}{2R^3}-\frac{1}{2}\eta_1a_1^2B_{12}+\eta_2a_2^2\Big(a_2^2A_{22}-\frac{1}{2}B_{22}\Big) \\
&\quad -\frac{1}{2}\eta_3a_3^2B_{23}-\frac{1}{2\pi\rho R^3}\sum_l\eta_lI_{ll} \\
&\quad +\frac{M}{2R^3}(2a_2^4A_{22}+2a_1^2B_{12}-a_2^2B_{22}-a_3^2B_{23}) \\
&\quad -\frac{1}{2}\Omega^2\Big\{\frac{R^2}{4}\Big(3a_2^2A_{22}-2A_2-\frac{P_0}{\pi\rho^2a_2^2}\Big)+F_a^2\Big(3A_0+a_1^2A_1-7a_2^2A_2 \\
&\qquad\qquad +4a_2^4A_{22}-2a_1^2a_2^2A_{12}-2a_1^2B_{12}-2a_2^2B_{22}+\frac{P_0}{\pi\rho^2}\Big)\Big\}\Big] \\
& +(F_a+1)\Omega p+O(R^{-4}),
\end{aligned}
\tag{2.83}
$$

$$
\begin{aligned}
\gamma_3 = {}& \pi\rho\Big[(\alpha_0-\pi\rho A_0)A_3+\frac{P_0}{\rho a_3^2}(A_0+a_3^2A_3)-\frac{1}{a_3^2}\Big(a_3^2A_3-\frac{P_0}{\pi\rho^2}\Big)\Big(\frac{M}{R}+\frac{3\not{I}_{11}}{2R^3}\Big) \\
&\quad +\Big(\frac{\alpha_0}{\pi\rho}+\frac{P_0}{\pi\rho^2}-A_0\Big)\frac{M}{2R^3}-\frac{1}{2}\eta_1a_1^2B_{13}-\frac{1}{2}\eta_2a_2^2B_{23} \\
&\quad +\eta_3a_3^2\Big(a_3^2A_{33}-\frac{1}{2}B_{33}\Big)-\frac{1}{2\pi\rho R^3}\sum_l\eta_lI_{ll}
\end{aligned}
$$

$$
\begin{aligned}
&\quad +\frac{M}{2R^3}(2a_3^4A_{33}+2a_1^2B_{13}-a_2^2B_{23}-a_3^2B_{33})\\
&\quad -\frac{1}{2}\Omega^2\Big\{\frac{R^2}{4}\Big(a_2^2A_{23}-\frac{P_0}{\pi\rho^2a_3^2}\Big)-2F_a^2(a_1^2B_{13}+a_2^2B_{23})\Big\}\Big]\\
&+O(R^{-4}), \qquad (2.84)\\
\gamma_{11} &= \pi\rho\Big[-\pi\rho\frac{A_1}{a_1^2}\Big(\frac{P_0}{\pi\rho^2}-\frac{1}{2}a_1^2A_1\Big)-\frac{M}{a_1^2R^3}\Big(a_1^2A_1-\frac{P_0}{\pi\rho^2}\Big)\\
&\quad -\eta_1a_1^2\Big(a_1^2A_{111}-\frac{1}{4}B_{111}\Big)+\frac{1}{4}\eta_2a_2^2B_{112}+\frac{1}{4}\eta_3a_3^2B_{113}\\
&\quad +\frac{M}{4R^3}(8a_1^4A_{111}-2a_1^2B_{111}+a_2^2B_{112}+a_3^2B_{113})\\
&\quad -\frac{1}{2}\Omega^2F_a^2\Big(-3A_1+7a_1^2A_{11}-a_2^2A_{12}+2a_1^2a_2^2A_{112}-4a_1^4A_{111}\\
&\qquad\qquad +a_1^2B_{111}+a_2^2B_{112}-\frac{P_0}{\pi\rho^2a_1^2}\Big)+O(R^{-5})\Big]\\
&+(F_a-1)\Omega q+O(R^{-6}), \qquad (2.85)\\
\gamma_{12} &= \pi\rho\Big[-\frac{P_0}{\rho}\Big(\frac{A_2}{a_1^2}+\frac{A_1}{a_2^2}\Big)+\pi\rho A_1A_2-\frac{M}{2R^3}\Big(2A_2-A_1-\frac{2P_0}{\pi\rho^2a_2^2}+\frac{P_0}{\pi\rho^2a_1^2}\Big)\\
&\quad -\eta_1a_1^2\Big(a_1^2A_{112}-\frac{1}{2}B_{112}\Big)-\eta_2a_2^2\Big(a_2^2A_{122}-\frac{1}{2}B_{122}\Big)+\frac{1}{2}\eta_3a_3^2B_{123}\\
&\quad +\frac{M}{2R^3}(4a_1^4A_{112}-2a_2^4A_{122}-2a_1^2B_{112}+a_2^2B_{122}+a_3^2B_{123})\\
&\quad -\frac{1}{2}\Omega^2F_a^2\Big\{-3(A_1+A_2)+5(a_1^2+a_2^2)A_{12}-3(a_1^2A_{11}+a_2^2A_{22})\\
&\qquad\qquad +6a_1^2a_2^2(A_{112}+A_{122})-4a_1^4A_{112}-4a_2^4A_{122}+2a_1^2B_{112}\\
&\qquad\qquad +2a_2^2B_{122}-\frac{P_0}{\pi\rho^2}\Big(\frac{1}{a_1^2}+\frac{1}{a_2^2}\Big)\Big\}+O(R^{-5})\Big]\\
&+3\Omega\big\{(F_a+1)q+(F_a-1)r\big\}+O(R^{-6}), \qquad (2.86)\\
\gamma_{13} &= \pi\rho\Big[-\frac{P_0}{\rho}\Big(\frac{A_3}{a_1^2}+\frac{A_1}{a_3^2}\Big)+\pi\rho A_1A_3-\frac{M}{2R^3}\Big(2A_3-A_1-\frac{2P_0}{\pi\rho^2a_3^2}+\frac{P_0}{\pi\rho^2a_1^2}\Big)\\
&\quad -\eta_1a_1^2\Big(a_1^2A_{113}-\frac{1}{2}B_{113}\Big)+\frac{1}{2}\eta_2a_2^2B_{123}-\eta_3a_3^2\Big(a_3^2A_{133}-\frac{1}{2}B_{133}\Big)\\
&\quad +\frac{M}{2R^3}(4a_1^4A_{113}-2a_3^4A_{133}-2a_1^2B_{113}+a_2^2B_{123}+a_3^2B_{133})\\
&\quad -\frac{1}{2}\Omega^2F_a^2\Big(-3A_3+7a_1^2A_{13}-a_2^2A_{23}+2a_1^2a_2^2A_{123}-4a_1^4A_{113}\\
&\qquad\qquad +2a_1^2B_{113}+2a_2^2B_{123}-\frac{P_0}{\pi\rho^2a_3^2}\Big)+O(R^{-5})\Big]\\
&+(F_a-1)\Omega s+O(R^{-6}), \qquad (2.87)\\
\gamma_{22} &= \pi\rho\Big[-\pi\rho\frac{A_2}{a_2^2}\Big(\frac{P_0}{\pi\rho^2}-\frac{1}{2}a_2^2A_2\Big)-\frac{M}{2a_2^2R^3}\Big(\frac{P_0}{\pi\rho^2}-a_2^2A_2\Big)\\
&\quad +\frac{1}{4}\eta_1a_1^2B_{122}-\eta_2a_2^2\Big(a_2^2A_{222}-\frac{1}{4}B_{222}\Big)+\frac{1}{4}\eta_3a_3^2B_{223}\\
&\quad -\frac{M}{4R^3}(4a_2^4A_{222}+2a_1^2B_{122}-a_2^2B_{222}-a_3^2B_{223})\\
&\quad -\frac{1}{2}\Omega^2F_a^2\Big(-3A_2+7a_2^2A_{22}-a_1^2A_{12}+2a_1^2a_2^2A_{122}-4a_2^4A_{222}\\
&\qquad\qquad +a_1^2B_{122}+a_2^2B_{222}-\frac{P_0}{\pi\rho^2a_2^2}\Big)+O(R^{-5})\Big]\\
&+(F_a+1)\Omega r+O(R^{-6}), \qquad (2.88)
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{23} = \pi\rho\Big[&-\frac{P_0}{\rho}\Big(\frac{A_3}{a_2^2}+\frac{A_2}{a_3^2}\Big)+\pi\rho A_2A_3-\frac{M}{2R^3}\Big(-A_2-A_3+\frac{P_0}{\pi\rho^2a_2^2}+\frac{P_0}{\pi\rho^2a_3^2}\Big)\\
&+\frac{1}{2}\eta_1a_1^2B_{123}-\eta_2a_2^2\Big(a_2^2A_{223}-\frac{1}{2}B_{223}\Big)\\
&-\eta_3a_3^2\Big(a_3^2A_{233}-\frac{1}{2}B_{233}\Big)\\
&-\frac{M}{2R^3}(2a_2^4A_{223}+2a_3^4A_{233}+2a_1^2B_{123}-a_2^2B_{223}-a_3^2B_{233})\\
&-\frac{1}{2}\Omega^2F_a^2\Big(-3A_3+7a_2^2A_{23}-a_1^2A_{13}+2a_1^2a_2^2A_{123}-4a_2^4A_{223}\\
&\qquad\qquad+2a_1^2B_{123}+2a_2^2B_{223}-\frac{P_0}{\pi\rho^2a_3^2}\Big)+O(R^{-5})\Big]\\
&+(F_a+1)\Omega s+O(R^{-6}),
\end{aligned}
\tag{2.89}
$$

$$
\begin{aligned}
\gamma_{33} = \pi\rho\Big[&-\pi\rho\frac{A_3}{a_3^2}\Big(\frac{P_0}{\pi\rho^2}-\frac{1}{2}a_3^2A_3\Big)-\frac{M}{2a_3^2R^3}\Big(\frac{P_0}{\pi\rho^2}-a_3^2A_3\Big)\\
&+\frac{1}{4}\eta_1a_1^2B_{133}+\frac{1}{4}\eta_2a_2^2B_{233}-\eta_3a_3^2\Big(a_3^2A_{333}-\frac{1}{4}B_{333}\Big)\\
&-\frac{M}{4R^3}(4a_3^4A_{333}+2a_1^2B_{133}-a_2^2B_{233}-a_3^2B_{333})\\
&-\frac{1}{2}\Omega^2F_a^2(a_1^2B_{133}+a_2^2B_{233})+O(R^{-5})\Big],
\end{aligned}
\tag{2.90}
$$

$$
\begin{aligned}
\kappa_0 = \pi\rho\Big[&\Big(\frac{P_0}{\pi\rho^2}+\frac{\alpha_0}{\pi\rho}-A_0\Big)\Big(\frac{M}{R^2}+\frac{9\not{I}_{11}}{2R^4}\Big)+\alpha_1a_1^2A_1-\sum_l\frac{\eta_l}{\pi\rho}\Big(\frac{I_{ll}}{R^2}+\frac{9\not{I}_{ll11}}{2R^4}\Big)\\
&+\frac{3M}{4R^4}a_1^2(2a_1^2B_{11}-a_2^2B_{12}-a_3^2B_{13})\\
&-\frac{1}{4}\Omega^2F_aR\Big(-A_0+a_1^2A_1-2a_1^2a_2^2A_{12}+\frac{2P_0}{\pi\rho^2}\Big)+O(R^{-6})\Big]\\
&-\frac{M^2}{R^3}-\frac{3M}{R^5}\not{I}_{11}+\frac{2\alpha_1}{R^3}I_{11}\\
&-\frac{1}{8}\Omega^2\Big[2(13F_a-7)M-\frac{1}{R^2}\Big\{4F_a(11F_a-15)I_{11}+(12F_a^2+2F_a+3)I_{22}\\
&\qquad\qquad+3(21-13F_a)\not{I}_{11}\Big\}\Big]\\
&-\frac{1}{16}\Omega^4R^3F_a+(F_a-1)\Omega e+O(R^{-7}),
\end{aligned}
\tag{2.91}
$$

$$
\begin{aligned}
\kappa_1 = \pi\rho\Big[&\Big(\frac{P_0}{\pi\rho^2}+\frac{\alpha_0}{\pi\rho}-A_0\Big)\frac{M}{R^4}-\frac{1}{a_1^2}\Big(\frac{P_0}{\pi\rho^2}-a_1^2A_1\Big)\Big(\frac{M}{R^2}+\frac{9\not{I}_{11}}{2R^4}\Big)\\
&-\alpha_1a_1^2A_{11}-\frac{1}{\pi\rho R^4}\sum_l\eta_lI_{ll}\\
&+\frac{M}{2R^4}a_1^2(4a_1^4A_{111}-6a_1^2B_{111}+3a_2^2B_{112}+3a_3^2B_{113})\\
&-\frac{1}{4}\Omega^2RF_a\Big(A_1-a_1^2A_{11}+2a_1^2a_2^2A_{112}-\frac{2P_0}{\pi\rho^2a_1^2}\Big)+O(R^{-6})\Big]\\
&-\frac{2M^2}{R^5}+\frac{M}{4R^2}\Omega^2(6F_a^2-13F_a+7)-\frac{1}{4}\Omega^4RF_a^3+(F_a-1)\Omega f\\
&+O(R^{-7}),
\end{aligned}
\tag{2.92}
$$

$$
\begin{aligned}
\kappa_2 = \pi\rho\Big[&-\Big(\frac{P_0}{\pi\rho^2}+\frac{\alpha_0}{\pi\rho}-A_0\Big)\frac{3M}{2R^4}-\frac{1}{a_2^2}\Big(\frac{P_0}{\pi\rho^2}-a_2^2A_2\Big)\Big(\frac{M}{R^2}+\frac{9\not{I}_{11}}{2R^4}\Big)\\
&-\alpha_1a_1^2A_{12}+\frac{3}{2\pi\rho R^4}\sum_l\eta_lI_{ll}
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{3M}{2R^4}a_1^2(2a_2^4A_{122}+2a_1^2B_{112}-a_2^2B_{122}-a_3^2B_{123}) \\
& +\frac{1}{4}\Omega^2 RF_a\Big(2A_1+A_2+3a_1^2A_{12}-2a_2^2A_{12}-6a_1^2a_2^2A_{122}+\frac{2P_0}{\pi\rho^2a_2^2}\Big) \\
& +O(R^{-6})\Big] \\
& +\frac{2M^2}{R^5}+\frac{3M}{8R^2}\Omega^2(4F_a^2+5F_a-6)-\frac{1}{4}\Omega^4RF_a^3+2(F_a+1)\Omega f \\
& +3(F_a-1)\Omega g+O(R^{-7}),
\end{aligned}
\tag{2.93}
$$

$$
\begin{aligned}
\kappa_3 = {} & \pi\rho\Big[-\Big(\frac{P_0}{\pi\rho^2}+\frac{\alpha_0}{\pi\rho}-A_0\Big)\frac{3M}{2R^4}-\frac{1}{a_3^2}\Big(\frac{P_0}{\pi\rho^2}-a_3^2A_3\Big)\Big(\frac{M}{R^2}+\frac{9\not{I}_{11}}{2R^4}\Big) \\
& -\alpha_1a_1^2A_{13}+\frac{3}{2\pi\rho R^4}\sum_l \eta_l I_{ll} \\
& -\frac{3M}{2R^4}a_1^2(2a_3^4A_{133}+2a_1^2B_{113}-a_2^2B_{123}-a_3^2B_{133}) \\
& -\frac{1}{4}\Omega^2RF_a\Big(A_3-a_1^2A_{13}+2a_1^2a_2^2A_{123}-\frac{2P_0}{\pi\rho^2a_3^2}\Big)+O(R^{-6})\Big] \\
& +\frac{2M^2}{R^5}+\frac{M}{8R^2}\Omega^2(13F_a-21)+(F_a-1)\Omega h+O(R^{-7}),
\end{aligned}
\tag{2.94}
$$

$$
\begin{aligned}
\kappa_{11} = {} & \frac{M\pi\rho}{R^4}\Big[-\frac{P_0}{\pi\rho^2a_1^2}+A_1 \\
& -\frac{a_1^2}{4}(8a_1^4A_{1111}-6a_1^2B_{1111}+3a_2^2B_{1112}+3a_3^2B_{1113})\Big],
\end{aligned}
\tag{2.95}
$$

$$
\begin{aligned}
\kappa_{12} = {} & \frac{M\pi\rho}{2R^4}\Big[-\frac{P_0}{\pi\rho^2}\Big(\frac{2}{a_2^2}-\frac{3}{a_1^2}\Big)-3A_1+2A_2 \\
& -a_1^2(4a_1^4A_{1112}-6a_2^4A_{1122}-6a_1^2B_{1112}+3a_2^2B_{1122}+3a_3^2B_{1123})\Big],
\end{aligned}
\tag{2.96}
$$

$$
\begin{aligned}
\kappa_{13} = {} & \frac{M\pi\rho}{2R^4}\Big[-\frac{P_0}{\pi\rho^2}\Big(\frac{2}{a_3^2}-\frac{3}{a_1^2}\Big)-3A_1+2A_3 \\
& -a_1^2(4a_1^4A_{1113}-6a_3^4A_{1133}-6a_1^2B_{1113}+3a_2^2B_{1123}+3a_3^2B_{1133})\Big],
\end{aligned}
\tag{2.97}
$$

$$
\begin{aligned}
\kappa_{22} = {} & \frac{3M\pi\rho}{2R^4}\Big[\frac{P_0}{\pi\rho^2a_2^2}-A_2 \\
& +\frac{a_1^2}{2}(4a_2^4A_{1222}+2a_1^2B_{1122}-a_2^2B_{1222}-a_3^2B_{1223})\Big],
\end{aligned}
\tag{2.98}
$$

$$
\begin{aligned}
\kappa_{23} = {} & \frac{3M\pi\rho}{2R^4}\Big[\frac{P_0}{\pi\rho^2}\Big(\frac{1}{a_2^2}+\frac{1}{a_3^2}\Big)-A_2-A_3 \\
& +a_1^2(2a_2^4A_{1223}+2a_3^4A_{1233}+2a_1^2B_{1123}-a_2^2B_{1223}-a_3^2B_{1233})\Big],
\end{aligned}
\tag{2.99}
$$

$$
\begin{aligned}
\kappa_{33} = {} & \frac{3M\pi\rho}{2R^4}\Big[\frac{P_0}{\pi\rho^2a_3^2}-A_3 \\
& +\frac{a_1^2}{2}(4a_3^4A_{1333}+2a_1^2B_{1133}-a_2^2B_{1233}-a_3^2B_{1333})\Big].
\end{aligned}
\tag{2.100}
$$

The quantity δU denotes the gravitational potential induced by the deformation of the binary system. The explicit form of δU is given in Appendix A.

III. THE POST-NEWTONIAN ANGULAR VELOCITY

In this section, the 1PN correction for the orbital angular velocity at 1PN order is derived using the first tensor virial (TV) equation. The first TV relation is derived from

$$\int d^3x \frac{\partial P}{\partial x_1} = 0. \tag{3.1}$$

Substituting Eq. (2.80) into Eq. (3.1), we have Eq. (2.3) for Newtonian order. At 1PN order, the explicit form becomes

$$\begin{aligned} 0 = {} & \frac{MR}{2}\delta\Omega^2 + \delta\int d^3x\rho\frac{\partial U}{\partial x_1} + \Omega_{\rm DR}^2 F_a(2-F_a)\int d^3x\rho\xi_1 \\ & -\big(\kappa_0 M + 3\kappa_1 I_{11} + \kappa_2 I_{22} + \kappa_3 I_{33} + 5\kappa_{11} I_{1111} + 3\kappa_{12} I_{1122} + 3\kappa_{13} I_{1133} \\ & \quad + \kappa_{22} I_{2222} + \kappa_{23} I_{2233} + \kappa_{33} I_{3333}\big), \end{aligned} \tag{3.2}$$

where ξ_1 is the x_1 component of the Lagrangian displacement vectors ξ_i.

A. Deformation of the figure

The density profile of an incompressible and homogeneous sphere at 1PN order is the same as that at Newtonian order. However, the density profile of an ellipsoid at 1PN order is different from that at Newtonian order. Chandrasekhar's method to obtain the 1PN correction to the Newtonian figure is as follows. [27] First, the ellipsoidal figures at Newtonian order are constructed. Next, the 1PN effect is regarded as a small perturbation to the Newtonian configuration, and the deformation from the Newtonian ellipsoid is calculated by using the Lagrangian displacement vectors. Finally, by solving equations for the Lagrangian displacement and calculating the correction to U by the deformation, the equilibrium configuration at 1PN order is obtained. In this paper, we follow this method.

In choosing the Lagrangian displacement vectors $\xi_k^{(ij)}$, we require them to be divergent free $\sum_k \partial_k \xi_k^{(ij)} = 0$, due to incompressibility. To obtain $\delta\Omega^2$ up to $O(R^{-3}) \times \Omega_{\rm DR}^2$, we write the Lagrangian displacement vectors as

$$\xi_k = \frac{1}{c^2}\sum_{ij} S_{ij}\xi_k^{(ij)}, \tag{3.3}$$

where

$$\xi_k^{(11)} = (x_1,\ 0,\ -x_3), \tag{3.4}$$
$$\xi_k^{(12)} = (0,\ x_2,\ -x_3), \tag{3.5}$$
$$\xi_k^{(31)} = \Big(\frac{1}{3}x_1^3,\ -x_1^2 x_2,\ 0\Big), \tag{3.6}$$
$$\xi_k^{(32)} = \Big(0,\ \frac{1}{3}x_2^3,\ -x_2^2 x_3\Big), \tag{3.7}$$
$$\xi_k^{(33)} = \Big(-x_3^2 x_1,\ 0,\ \frac{1}{3}x_3^3\Big), \tag{3.8}$$
$$\xi_k^{(0)} = \Big(\frac{1}{2},\ 0,\ 0\Big), \tag{3.9}$$
$$\xi_k^{(21)} = \Big(\frac{1}{2}x_1^2,\ 0,\ -x_1 x_3\Big), \tag{3.10}$$
$$\xi_k^{(22)} = (0,\ x_1 x_2,\ -x_1 x_3). \tag{3.11}$$

Here, we consider only up to the biquadratic deformation so that the Lagrangian displacement vectors of higher order functions in x_i can be neglected. The Lagrangian displacement vectors (3.4) – (3.8) are concerned with the deformation of the star, i.e., coefficients of the velocity potential of the 1PN order p, q, r, and s. On the other hand, (3.9) – (3.11) are associated with e, f, g, and h.

In order to obtain the orbital angular velocity of 1PN order, we must calculate the second and third terms on the right-hand side of Eq. (3.2), which come from the displacement of the fluid element by the 1PN correction. The second term can be evaluated as follows: First, the contribution from star 1 is zero because

$$\begin{aligned} \delta\int d^3x\rho\frac{\partial U^{1\to 1}}{\partial x_1} &= -\delta\int d^3x\rho(\mathbf{x})\int d^3x'\rho(\mathbf{x}')\frac{x_1 - x_1'}{|\mathbf{x}-\mathbf{x}'|^3} \\ &= -\int d^3x\rho(\mathbf{x})\delta\int d^3x'\rho(\mathbf{x}')\frac{x_1 - x_1'}{|\mathbf{x}-\mathbf{x}'|^3} \end{aligned}$$

$$
\begin{aligned}
&+\int d^3x'\rho(\mathbf{x}')\delta\int d^3x\rho(\mathbf{x})\frac{x_1'-x_1}{|\mathbf{x}-\mathbf{x}'|^3}\\
&=0.
\end{aligned}
\tag{3.12}
$$

The contribution from star 2 can be separated into two parts as

$$
\delta\int d^3x\rho\frac{\partial U^{2\to 1}}{\partial x_1}=\int d^3x\rho\sum_{i=1}^{3}\xi_i\frac{\partial^2U^{2\to 1}}{\partial x_1\partial x_i}+\int d^3x\rho\frac{\partial\delta U^{2\to 1}}{\partial x_1}.
\tag{3.13}
$$

The first term on the right-hand side of Eq. (3.13) denotes the force to which the displaced element of star 1 is subject from the nondisplaced potential of star 2. On the other hand, the second term denotes the force to which the nondisplaced element of star 1 is subject from the displaced potential of star 2. For the Lagrangian displacement vectors given in Eqs. (3.4) – (3.11), the first and the second terms on the right-hand side of Eq. (3.13) are equal, and we can calculate as

$$
\begin{aligned}
\delta\int d^3x\rho\frac{\partial U}{\partial x_1}&=2\int d^3x\rho\sum_{i=1}^{3}\xi_i\frac{\partial^2U^{2\to 1}}{\partial x_1\partial x_i}\\
&=2\int d^3x\rho\frac{\partial\delta U^{2\to 1}}{\partial x_1}\\
&=2\Big[-\frac{M}{R^4}\Big\{3S_{11}(2I_{11}+I_{33})+3S_{12}(I_{33}-I_{22})+S_{31}(2I_{1111}+3I_{1122})\\
&\qquad\qquad+S_{32}(3I_{2233}-I_{2222})-S_{33}(I_{3333}+6I_{1133})\Big\}\\
&\quad+S_0M\Big(\frac{M}{R^3}+\frac{18\bar{I}_{11}}{R^5}\Big)\\
&\quad+S_{21}\frac{M}{R^3}\Big(I_{11}+\frac{9\bar{I}_{1111}}{R^2}+\frac{9\bar{I}_{11}I_{11}}{MR^2}+\frac{12I_{1133}}{R^2}\Big)\\
&\quad+S_{22}\frac{12M}{R^5}(I_{1133}-I_{1122})\Big].
\end{aligned}
\tag{3.14}
$$

Also, the third term on the right-hand side of Eq. (3.2) is written as

$$
\Omega_{\rm DR}^2F_a(2-F_a)\int d^3x\rho\xi_1=\frac{1}{2}\Omega_{\rm DR}^2F_a(2-F_a)(MS_0+S_{21}I_{11}).
\tag{3.15}
$$

B. The post-Newtonian orbital angular velocity

The angular velocity at 1PN order is calculated up to $O(R^{-6})$ from Eq. (3.2) as

$$
\begin{aligned}
\delta\Omega^2=&\frac{4}{R^5}\Big[3S_{11}(2I_{11}+I_{33})+3S_{12}(I_{33}-I_{22})+S_{31}(2I_{1111}+3I_{1122})\\
&\qquad+S_{32}(3I_{2233}-I_{2222})-S_{33}(I_{3333}+6I_{1133})\Big]\\
&-\frac{2S_0}{R}\Big[\frac{2M}{R^3}+\frac{36\bar{I}_{11}}{R^5}+\frac{1}{2}\Omega_{\rm DR}^2F_a(2-F_a)\Big]\\
&-\frac{2S_{21}}{R}\Big[\frac{2I_{11}}{R^3}+\frac{18\bar{I}_{11}}{5R^5}a_1^2+\frac{18\bar{I}_{1111}}{R^5}+\frac{24I_{1133}}{R^5}+\frac{1}{10}\Omega_{\rm DR}^2F_a(2-F_a)a_1^2\Big]\\
&-\frac{48}{R^6}S_{22}(I_{1133}-I_{1122})\\
&+2\pi\rho\Big[\Big(\frac{\alpha_0}{\pi\rho}-\frac{4}{5}A_0+\frac{2}{5}a_1^2A_1\Big)\Big(\frac{M}{R^3}+\frac{9\bar{I}_{11}}{2R^5}\Big)+\frac{2\alpha_1}{5R}a_1^2A_1\\
&\qquad+\frac{9\bar{I}_{11}}{2R^5}\Big(\frac{P_0}{\pi\rho^2}+\frac{\alpha_0}{\pi\rho}-A_0\Big)-\sum_l\frac{\eta_l}{\pi\rho}\Big(\frac{I_{ll}}{R^3}+\frac{9\bar{I}_{ll11}}{2R^5}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad -\frac{9\rlap{-}I_{11}}{2\pi\rho M R^5}\sum_l \eta_l I_{ll} + \frac{1}{5}\Omega_{\rm DR}^2 F_a (2a_2^2 A_2 + a_3^2 A_3) \\
&\qquad\qquad +\frac{3M}{70R^5}\Big\{\Big(-\frac{7P_0}{\pi\rho^2} + A_0 + 6a_1^2 A_1\Big)(2a_1^2 - a_2^2 - a_3^2) \\
&\qquad\qquad\qquad\qquad +2(2a_1^4 A_1 - a_2^4 A_2 - a_3^4 A_3)\Big\}\Big] \\
&-\frac{2M^2}{R^4} - \frac{2M}{R^6}(8I_{11} - 3I_{22} - 3I_{33}) + \frac{4\alpha_1}{R^4} I_{11} \\
&-\frac{1}{8}\Omega_{\rm DR}^4 R^2 F_a \Big\{1 + \frac{4}{5R^2}(3a_1^2 + a_2^2) F_a^2\Big\} \\
&-\frac{1}{2R}\Omega_{\rm DR}^2\Big\{(13F_a - 7)M \\
&\qquad -\frac{1}{R^2}\Big(2(20F_a^2 - 41F_a + 21)I_{11} + 3(4F_a^2 + 5F_a - 6)I_{22} \\
&\qquad\qquad +(13F_a - 21)I_{33}\Big)\Big\} \\
&+(F_a - 1)\frac{2\Omega_{\rm DR}}{R}\Big(e + \frac{3}{5}a_1^2 f + \frac{3}{5}a_2^2 g + \frac{1}{5}a_3^2 h\Big) + \frac{4a_2^2}{5R}(F_a + 1)\Omega_{\rm DR} f \\
&+O(R^{-7}), \qquad\qquad (3.16) \\
=\ &\frac{4}{R^5}\Big[3S_{11}(2I_{11} + I_{33}) + 3S_{12}(I_{33} - I_{22}) + S_{31}(2I_{1111} + 3I_{1122}) \\
&\qquad +S_{32}(3I_{2233} - I_{2222}) - S_{33}(I_{3333} + 6I_{1133})\Big] \\
&-\frac{2S_0}{R}\Big[\frac{2M}{R^3} + \frac{36\rlap{-}I_{11}}{R^5} + \frac{1}{2}\Omega_{\rm DR}^2 F_a (2 - F_a)\Big] \\
&-\frac{2S_{21}}{R}\Big[\frac{2I_{11}}{R^3} + \frac{18\rlap{-}I_{11}}{5R^5}a_1^2 + \frac{18\rlap{-}I_{1111}}{R^5} + \frac{24I_{1133}}{R^5} + \frac{1}{10}\Omega_{\rm DR}^2 F_a (2 - F_a) a_1^2\Big] \\
&-\frac{48}{R^6}S_{22}(I_{1133} - I_{1122}) \\
&+(F_a - 1)\frac{2\Omega_{\rm DR}}{R}\Big(e + \frac{3}{5}a_1^2 f + \frac{3}{5}a_2^2 g + \frac{1}{5}a_3^2 h\Big) + \frac{4a_2^2}{5R}(F_a + 1)\Omega_{\rm DR} f \\
&+\frac{4M\pi\rho}{5R^3}(4 + F_a)A_0 + \frac{9M^2}{2R^4}(2 - 3F_a) + \frac{12\pi\rho}{5R^5}(11 + 3F_a)A_0 \rlap{-}I_{11} \\
&+\frac{M}{R^6}\Big[2(-3F_a^3 + 12F_a^2 - 77F_a + 54)I_{11} + (-10F_a^3 + 63F_a - 43)I_{22} \\
&\qquad +(55F_a - 46)I_{33}\Big] + O(R^{-7}). \qquad\qquad (3.17)
\end{aligned}
$$

IV. BOUNDARY CONDITIONS

In this section, we derive equations to determine the coefficients of the 1PN velocity potential and the deformation of the binary system, p, q, r, s, e, f, g, h, and S_{ij}, from the boundary conditions on the stellar surface. The stellar surfaces of the Darwin-Riemann ellipsoid and its deformed figure are expressed as $S_{\rm DR}(x) = 0$ and $S(x) = 0$, respectively, where

$$S_{\rm DR}(x) = \sum_l \frac{x_l^2}{a_l^2} - 1, \qquad (4.1)$$

$$
\begin{aligned}
S(x) &= \sum_l \frac{x_l^2}{a_l^2} - 1 - \sum_j \xi_j \frac{\partial S_{\rm DR}}{\partial x_j}, \qquad (4.2) \\
&= S_{\rm DR}(x) - \frac{2}{c^2}\Big[S_{11}\Big(\frac{x_1^2}{a_1^2} - \frac{x_3^2}{a_3^2}\Big) + S_{12}\Big(\frac{x_2^2}{a_2^2} - \frac{x_3^2}{a_3^2}\Big) + S_{31}\Big(\frac{x_1^4}{3a_1^2} - \frac{x_1^2 x_2^2}{a_2^2}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+S_{32}\Big(\frac{x_2^4}{3a_2^2}-\frac{x_2^2x_3^2}{a_3^2}\Big)+S_{33}\Big(\frac{x_3^4}{3a_3^2}-\frac{x_3^2x_1^2}{a_1^2}\Big)\\
&+\frac{1}{2}S_0\frac{x_1}{a_1^2}+S_{21}\Big(\frac{x_1^3}{2a_1^2}-\frac{x_1x_3^2}{a_3^2}\Big)\\
&+S_{22}\Big(\frac{x_1x_2^2}{a_2^2}-\frac{x_1x_3^2}{a_3^2}\Big)\Big]. \qquad (4.3)
\end{aligned}
$$

The boundary conditions for the continuity equation and integrated Euler equation are, respectively,

$$(A)\quad C_i\frac{\partial S}{\partial x_i}=0 \quad \text{on} \quad S=0, \qquad (4.4)$$

$$(B)\quad \frac{P}{\rho}=0 \quad \text{on} \quad S=0. \qquad (4.5)$$

The condition (A) comes from the constraint that the normal component of the velocity on the surface vanishes. The condition (B) determines the stellar surface. Equations (4.4) and (4.5) must hold to order c^{-2}.

A. Condition (A)

Substituting C_i given by Eq. (2.14) into Eq. (4.4), we obtain the equation at 1PN order

$$
\begin{aligned}
0=&\frac{2\Omega x_1x_2}{c^2}\big(\sigma_0+\sigma_1x_1^2+\sigma_2x^2+\sigma_3x_3^2\big)\\
&+\frac{2\Omega x_2}{c^2}\big(\lambda_0+\lambda_1x_1^2+\lambda_2x_2^2+\lambda_3x_3^2\big), \qquad (4.6)
\end{aligned}
$$

where σ_k $(k=0,1,2,3)$ are functions of a_k, p, q, r, s, and S_{ij}, and also λ_k $(k=0,1,2,3)$ are functions of a_k, e, f, g, h, and S_{ij}. Equation (4.6) must be satisfied on $S_{\rm DR}$. Then, we obtain the following six equations (three equations for p, q, r, s, and S_{ij}, and three equations for e, f, g, h, and S_{ij}):

$$
\begin{aligned}
\Sigma_1^{(A)}=&\frac{a_1^2+a_2^2}{a_1^4a_2^2}p+\frac{a_1^2+3a_2^2}{a_1^2a_2^2}q+\frac{\Omega}{a_1^2+a_2^2}\Big[\frac{4}{a_1^2}(-S_{11}+S_{12})-\frac{20}{3}S_{31}\Big]\\
&-3\pi\rho\Omega(A_0-a_1^2A_1)\frac{a_1^2-a_2^2}{a_1^4a_2^2}-\frac{F_a^2\Omega^3}{2}\Big(\frac{a_1^2-a_2^2}{a_1^2a_2^2}\Big)\\
&+\frac{\pi\rho F_a\Omega}{2a_1^2}\Big[\Big(\frac{7a_1^2}{a_2^2}-1\Big)B_{11}+\Big(\frac{7a_2^2}{a_1^2}-1\Big)B_{12}+4(a_1^2+a_2^2)B_{112}\Big]\\
&-\frac{R^2\Omega^3}{8}\frac{3a_1^2+a_2^2}{a_1^4a_2^2}F_a+\frac{M\Omega}{4R}\Big(\frac{a_1^2+11a_2^2}{a_1^4a_2^2}\Big)\\
&+\frac{\Omega}{2R^3a_1^4}\Big(\frac{9M}{2}a_1^2+2F_aI_{11}+10F_aI_{22}+\frac{3}{2}I_{22}+\frac{27}{4}\not{I}_{11}\Big)\\
&-\frac{\Omega}{2R^3a_1^2a_2^2}\Big(-\frac{M}{2}a_1^2+14F_aI_{11}-2F_aI_{22}-\frac{3}{2}I_{22}-\frac{81}{4}\not{I}_{11}\Big)=0, \qquad (4.7)\\
\Sigma_2^{(A)}=&\frac{a_1^2+a_2^2}{a_1^2a_2^4}p+\frac{3a_1^2+a_2^2}{a_1^2a_2^2}r+\frac{\Omega}{a_1^2+a_2^2}\Big[\frac{4}{a_2^2}(-S_{11}+S_{12})+\frac{4a_1^2}{a_2^2}S_{31}+\frac{8}{3}S_{32}\Big]\\
&-3\pi\rho\Omega(A_0-a_2^2A_2)\frac{a_1^2-a_2^2}{a_1^2a_2^4}-\frac{F_a^2\Omega^3}{2}\Big(\frac{a_1^2-a_2^2}{a_1^2a_2^2}\Big)\\
&+\frac{\pi\rho F_a\Omega}{2a_2^2}\Big[\Big(\frac{7a_1^2}{a_2^2}-1\Big)B_{12}+\Big(\frac{7a_2^2}{a_1^2}-1\Big)B_{22}+4(a_1^2+a_2^2)B_{122}\Big]\\
&-\frac{R^2\Omega^3}{8}\frac{3a_1^2+a_2^2}{a_1^2a_2^4}F_a+\frac{M\Omega}{4R}\Big(\frac{a_1^2+11a_2^2}{a_1^2a_2^4}\Big)\\
&+\frac{\Omega}{2R^3a_1^2a_2^2}\Big(-\frac{9M}{4}a_2^2+2F_aI_{11}+10F_aI_{22}+\frac{3}{2}I_{22}+\frac{27}{4}\not{I}_{11}\Big)
\end{aligned}
$$

$$
-\frac{\Omega}{2R^3a_2^4}\Big(\frac{21M}{4}a_2^2+14F_aI_{11}-2F_aI_{22}-\frac{3}{2}I_{22}-\frac{81}{4}\not{I}_{11}\Big)=0, \tag{4.8}
$$

$$
\begin{aligned}
\Sigma_3^{(A)} = {} & \frac{a_1^2+a_2^2}{a_1^2a_2^2a_3^2}p+\Big(\frac{1}{a_1^2}+\frac{1}{a_2^2}+\frac{2}{a_3^2}\Big)s+\frac{\Omega}{a_1^2+a_2^2}\Big[\frac{4}{a_3^2}(-S_{11}+S_{12})-\frac{4a_2^2}{a_3^2}S_{32}+4S_{33}\Big] \\
& -3\pi\rho\Omega(A_0-a_3^2A_3)\frac{a_1^2-a_2^2}{a_1^2a_2^2a_3^2} \\
& +\frac{\pi\rho F_a\Omega}{2a_3^2}\Big[\Big(\frac{7a_1^2}{a_2^2}-1\Big)B_{13}+\Big(\frac{7a_2^2}{a_1^2}-1\Big)B_{23}+4(a_1^2+a_2^2)B_{123}\Big] \\
& -\frac{R^2\Omega^3}{8}\frac{3a_1^2+a_2^2}{a_1^2a_2^2a_3^2}F_a+\frac{M\Omega}{4R}\Big(\frac{a_1^2+11a_2^2}{a_1^2a_2^2a_3^2}\Big) \\
& +\frac{\Omega}{2R^3a_1^2a_3^2}\Big(-\frac{9M}{4}a_3^2+2F_aI_{11}+10F_aI_{22}+\frac{3}{2}I_{22}+\frac{27}{4}\not{I}_{11}\Big) \\
& -\frac{\Omega}{2R^3a_2^2a_3^2}\Big(-\frac{3M}{2}a_2^2+\frac{27M}{4}a_3^2+14F_aI_{11}-2F_aI_{22}-\frac{3}{2}I_{22}-\frac{81}{4}\not{I}_{11}\Big)=0,
\end{aligned} \tag{4.9}
$$

$$
\begin{aligned}
\Lambda_1^{(A)} = {} & \frac{e}{a_1^2a_2^2}+\frac{a_1^2+2a_2^2}{a_1^2a_2^2}f+\frac{\Omega}{a_1^2+a_2^2}\Big[-\frac{S_0}{a_1^2}-3S_{21}+4S_{22}\Big] \\
& +\frac{R\pi\rho\Omega}{4a_1^2a_2^2}(A_0-a_1^2A_1+3a_2^2B_{12})-\frac{R}{4}F_a^2\Omega^3\Big(\frac{3a_1^2+2a_2^2}{a_1^2a_2^2}\Big) \\
& -\frac{R^3\Omega^3}{16a_1^2a_2^2}-\frac{13M\Omega}{4a_1^2a_2^2} \\
& -\frac{\Omega}{2R^2a_1^2a_2^2}\Big(\frac{M}{2}a_1^2+5Ma_2^2-7F_aI_{11}+F_aI_{22}+\frac{1}{2}I_{22}+\frac{39}{4}\not{I}_{11}\Big) \\
& +O(R^{-4})=0,
\end{aligned} \tag{4.10}
$$

$$
\begin{aligned}
\Lambda_2^{(A)} = {} & \frac{e}{a_2^4}+\frac{3g}{a_2^2}-\frac{\Omega}{a_1^2+a_2^2}\Big[\frac{S_0}{a_2^2}+\frac{2a_1^2}{a_2^2}S_{22}\Big]+\frac{R\pi\rho\Omega}{4a_2^4}(A_0-a_2^2A_2+3a_2^2B_{22}) \\
& -\frac{RF_a^2\Omega^3}{4a_2^2}-\frac{R^3\Omega^3}{16a_2^4}-\frac{13M\Omega}{4a_2^4} \\
& +\frac{\Omega}{2R^2a_2^4}\Big(\frac{11M}{4}a_2^2+7F_aI_{11}-F_aI_{22}-\frac{1}{2}I_{22}-\frac{39}{4}\not{I}_{11}\Big) \\
& +O(R^{-4})=0,
\end{aligned} \tag{4.11}
$$

$$
\begin{aligned}
\Lambda_3^{(A)} = {} & \frac{e}{a_2^2a_3^2}+\frac{2a_2^2+a_3^2}{a_2^2a_3^2}h+\frac{\Omega}{a_1^2+a_2^2}\Big[-\frac{S_0}{a_3^2}+\frac{2a_1^2}{a_3^2}(S_{21}+S_{22})\Big] \\
& +\frac{R\pi\rho\Omega}{4a_2^2a_3^2}(A_0-a_3^2A_3+3a_2^2B_{23})-\frac{R^3\Omega^3}{16a_2^2a_3^2}-\frac{13M\Omega}{4a_2^2a_3^2} \\
& +\frac{\Omega}{4R^2a_2^2a_3^2}\Big(-Ma_2^2+\frac{13M}{2}a_3^2+14F_aI_{11}-2F_aI_{22}-I_{22}-\frac{39\not{I}_{11}}{2}\Big) \\
& +O(R^{-4})=0.
\end{aligned} \tag{4.12}
$$

We can derive the following two equations which do not contain S_{11}, S_{12} and p:

$$
\Sigma_1^{(A)}a_1^2-\Sigma_2^{(A)}a_2^2=0, \tag{4.13}
$$

$$
\Sigma_1^{(A)}a_1^2-\Sigma_3^{(A)}a_3^2=0. \tag{4.14}
$$

Thus, Eqs. (4.13) and (4.14) depend only on the coefficients q, r, s, S_{31}, S_{32} and S_{33}, i.e., the coefficients for biquadratic deformation. Also, we can construct two equations which do not contain e and S_0 as

$$
\Lambda_1^{(A)}a_1^2-\Lambda_2^{(A)}a_2^2=0, \tag{4.15}
$$

$$
\Lambda_1^{(A)}a_1^2-\Lambda_3^{(A)}a_3^2=0. \tag{4.16}
$$

Thus, Eqs. (4.15) and (4.16) depend only on the coefficients f, g, h, S_{21} and S_{22}, i.e., the coefficients for cubic deformation. The latter feature is important in determining coefficients.

B. Condition (B)

Substituting δU, which is given in Appendix A, and the surface equation $S=0$ into Eq. (2.80), we obtain

$$
\begin{aligned}
\Big(\frac{P}{\rho}\Big)_S = & -\frac{2P_0}{\rho c^2}\Big[S_{11}\Big(\frac{x_1^2}{a_1^2}-\frac{x_3^2}{a_3^2}\Big)+S_{12}\Big(\frac{x_2^2}{a_2^2}-\frac{x_3^2}{a_3^2}\Big)+S_{31}\Big(\frac{x_1^4}{3a_1^2}-\frac{x_1^2x_2^2}{a_2^2}\Big)\\
&\qquad +S_{32}\Big(\frac{x_2^4}{3a_2^2}-\frac{x_2^2x_3^2}{a_3^2}\Big)+S_{33}\Big(\frac{x_3^4}{3a_3^2}-\frac{x_3^2x_1^2}{a_1^2}\Big)+S_0\frac{x_1}{2a_1^2}\\
&\qquad +S_{21}\Big(\frac{x_1^3}{2a_1^2}-\frac{x_1x_3^2}{a_3^2}\Big)+S_{22}\Big(\frac{x_1x_2^2}{a_2^2}-\frac{x_1x_3^2}{a_3^2}\Big)\Big]\\
& +\frac{1}{c^2}\Big[S_{11}(D_{3,3}-D_{1,1})+S_{12}(D_{3,3}-D_{2,2})+S_{31}\Big(D_{112,2}-\frac{1}{3}D_{111,1}\Big)\\
&\qquad +S_{32}\Big(D_{223,3}-\frac{1}{3}D_{222,2}\Big)+S_{33}\Big(D_{133,1}-\frac{1}{3}D_{333,3}\Big)-\frac{1}{2}S_0U_{,1}^{1\to 1}\\
&\qquad +S_{21}\Big(D_{13,3}-\frac{1}{2}D_{11,1}\Big)+S_{22}(D_{13,3}-D_{12,2})\Big]\\
& +\frac{1}{c^2}\Big[S_{11}(D_{3,3}^{2\to 1}-D_{1,1}^{2\to 1})+S_{12}(D_{3,3}^{2\to 1}-D_{2,2}^{2\to 1})\\
&\qquad +S_{31}\Big(D_{112,2}^{2\to 1}-\frac{1}{3}D_{111,1}^{2\to 1}\Big)+S_{32}\Big(D_{223,3}^{2\to 1}-\frac{1}{3}D_{222,2}^{2\to 1}\Big)\\
&\qquad +S_{33}\Big(D_{133,1}^{2\to 1}-\frac{1}{3}D_{333,3}^{2\to 1}\Big)+\frac{1}{2}S_0U_{,1}^{2\to 1}\\
&\qquad -S_{21}\Big(D_{13,3}^{2\to 1}-\frac{1}{2}D_{11,1}^{2\to 1}\Big)-S_{22}(D_{13,3}^{2\to 1}-D_{12,2}^{2\to 1})\Big]\\
& +\frac{1}{2c^2}\delta\Omega^2\Big[F_a(2-F_a)x_1^2-F_a(2+F_a)x_2^2+Rx_1+\frac{R^2}{4}\Big]\\
& -\frac{1}{c^2}\Big[\gamma_0+\sum_l\gamma_l x_l^2+\sum_{l\le m}\gamma_{lm}x_l^2x_m^2\\
&\qquad +x_1\big(\kappa_0+\sum_l\kappa_l x_l^2+\sum_{l\le m}\kappa_{lm}x_l^2x_m^2\big)\Big]+\text{const}\\
\equiv & \ \frac{1}{c^2}\Big[Q_1x_1^2+Q_2x_2^2+Q_3x_3^2+Q_{11}x_1^4+Q_{22}x_2^4+Q_{33}x_3^4+Q_{12}x_1^2x_2^2\\
&\qquad +Q_{13}x_1^2x_3^2+Q_{23}x_2^2x_3^2\\
&\qquad +x_1\big(R_0+R_1x_1^2+R_2x_2^2+R_3x_3^2\big)+\text{const}\Big],
\end{aligned}
\tag{4.17}
$$

where

$$
\begin{aligned}
Q_1 = & -\gamma_1+\frac{1}{2}F_a(2-F_a)\delta\Omega^2+S_{11}\Big[-\pi\rho\Big(\frac{2P_0}{\pi\rho^2a_1^2}-3a_1^2A_{11}+a_3^2A_{13}\Big)+O(R^{-5})\Big]\\
& +S_{12}\Big[\pi\rho(a_2^2A_{12}-a_3^2A_{13})+O(R^{-5})\Big]\\
& +S_{31}\Big[\pi\rho a_1^2\Big(-a_1^4A_{111}+a_1^2a_2^2A_{112}+\frac{3}{2}a_1^2B_{111}-\frac{1}{2}a_2^2B_{112}\Big)+O(R^{-5})\Big]\\
& +S_{32}\Big[\frac{1}{2}\pi\rho a_2^2(a_2^2B_{122}-a_3^2B_{123})+O(R^{-5})\Big]\\
& +S_{33}\Big[\frac{1}{2}\pi\rho a_3^2(a_3^2B_{133}-3a_1^2B_{113})+O(R^{-5})\Big]+O(R^{-4})\times(S_0,\ S_{21}),
\end{aligned}
\tag{4.18}
$$

$$
Q_2 = -\gamma_2-\frac{1}{2}F_a(2+F_a)\delta\Omega^2+S_{11}\Big[\pi\rho(a_1^2A_{12}-a_3^2A_{23})+O(R^{-5})\Big]
$$

$$
\begin{aligned}
&+S_{12}\Big[-\pi\rho\Big(\frac{2P_0}{\pi\rho^2 a_2^2}-3a_2^2A_{22}+a_3^2A_{23}\Big)+O(R^{-5})\Big]\\
&+S_{31}\Big[\frac{1}{2}\pi\rho a_1^2(a_1^2B_{112}-3a_2^2B_{122})+O(R^{-5})\Big]\\
&+S_{32}\Big[\pi\rho a_2^2\Big(-a_2^4A_{222}+a_2^2a_3^2A_{223}+\frac{3}{2}a_2^2B_{222}-\frac{1}{2}a_3^2B_{223}\Big)+O(R^{-5})\Big]\\
&+S_{33}\Big[\frac{1}{2}\pi\rho a_3^2(a_3^2B_{233}-a_1^2B_{123})+O(R^{-5})\Big]+O(R^{-4})\times(S_0,\ S_{21}),
\end{aligned}
\tag{4.19}
$$

$$
\begin{aligned}
Q_3=&-\gamma_3+S_{11}\Big[\pi\rho\Big(\frac{2P_0}{\pi\rho^2 a_3^2}-3a_3^2A_{33}+a_1^2A_{13}\Big)+O(R^{-5})\Big]\\
&+S_{12}\Big[\pi\rho\Big(\frac{2P_0}{\pi\rho^2 a_3^2}-3a_3^2A_{33}+a_2^2A_{23}\Big)+O(R^{-5})\Big]\\
&+S_{31}\Big[\frac{1}{2}\pi\rho a_1^2(a_1^2B_{113}-a_2^2B_{123})+O(R^{-5})\Big]\\
&+S_{32}\Big[\frac{1}{2}\pi\rho a_2^2(a_2^2B_{223}-3a_3^2B_{233})+O(R^{-5})\Big]\\
&+S_{33}\Big[\pi\rho a_3^2\Big(-a_3^4A_{333}+a_3^2a_1^2A_{133}+\frac{3}{2}a_3^2B_{333}-\frac{1}{2}a_1^2B_{133}\Big)+O(R^{-5})\Big]\\
&+O(R^{-4})\times(S_0,\ S_{21}),
\end{aligned}
\tag{4.20}
$$

$$
\begin{aligned}
Q_{11}=&-\gamma_{11}+S_{31}\pi\rho\Big[-\frac{2P_0}{3\pi\rho^2 a_1^2}+\frac{5}{3}a_1^6A_{1111}-a_1^4a_2^2A_{1112}-\frac{5}{4}a_1^4B_{1111}+\frac{1}{4}a_1^2a_2^2B_{1112}\Big]\\
&+\frac{1}{4}S_{32}\pi\rho a_2^2(a_3^2B_{1123}-a_2^2B_{1122})+\frac{1}{4}S_{33}\pi\rho a_3^2(5a_1^2B_{1113}-a_3^2B_{1133}),
\end{aligned}
\tag{4.21}
$$

$$
\begin{aligned}
Q_{12}=&-\gamma_{12}+S_{31}\pi\rho\Big[\frac{2P_0}{\pi\rho^2 a_2^2}+a_1^6A_{1112}-3a_1^4a_2^2A_{1122}-\frac{3}{2}a_1^4B_{1112}+\frac{3}{2}a_1^2a_2^2B_{1122}\Big]\\
&+S_{32}\pi\rho a_2^2\Big(a_2^4A_{1222}-a_2^2a_3^2A_{1223}-\frac{3}{2}a_2^2B_{1222}+\frac{1}{2}a_3^2B_{1223}\Big)\\
&+\frac{1}{2}S_{33}\pi\rho a_3^2(3a_1^2B_{1123}-a_3^2B_{1233}),
\end{aligned}
\tag{4.22}
$$

$$
\begin{aligned}
Q_{13}=&-\gamma_{13}+S_{31}\pi\rho a_1^2\Big(a_1^4A_{1113}-a_1^2a_2^2A_{1123}-\frac{3}{2}a_1^2B_{1113}+\frac{1}{2}a_2^2B_{1123}\Big)\\
&+\frac{1}{2}S_{32}\pi\rho a_2^2(3a_3^2B_{1233}-a_2^2B_{1223})\\
&+S_{33}\pi\rho\Big[\frac{2P_0}{\pi\rho^2 a_1^2}+a_3^6A_{1333}-3a_3^4a_1^2A_{1133}+\frac{3}{2}a_1^2a_3^2B_{1133}-\frac{3}{2}a_3^4B_{1333}\Big],
\end{aligned}
\tag{4.23}
$$

$$
\begin{aligned}
Q_{22}=&-\gamma_{22}+\frac{1}{4}S_{31}\pi\rho a_1^2(5a_2^2B_{1222}-a_1^2B_{1122})\\
&+S_{32}\pi\rho\Big[-\frac{2P_0}{3\pi\rho^2 a_2^2}+\frac{5}{3}a_2^6A_{2222}-a_2^4a_3^2A_{2223}-\frac{5}{4}a_2^4B_{2222}+\frac{1}{4}a_2^2a_3^2B_{2223}\Big]\\
&+\frac{1}{4}S_{33}\pi\rho a_3^2(a_1^2B_{1223}-a_3^2B_{2233}),
\end{aligned}
\tag{4.24}
$$

$$
\begin{aligned}
Q_{23}=&-\gamma_{23}+\frac{1}{2}S_{31}\pi\rho a_1^2(3a_2^2B_{1223}-a_1^2B_{1123})\\
&+S_{32}\pi\rho\Big[\frac{2P_0}{\pi\rho^2 a_3^2}+a_2^6A_{2223}-3a_2^4a_3^2A_{2233}+\frac{3}{2}a_2^2a_3^2B_{2233}-\frac{3}{2}a_2^4B_{2223}\Big]\\
&+S_{33}\pi\rho a_3^2\Big(a_3^4A_{2333}-a_3^2a_1^2A_{1233}+\frac{1}{2}a_1^2B_{1233}-\frac{3}{2}a_3^2B_{2333}\Big),
\end{aligned}
\tag{4.25}
$$

$$
\begin{aligned}
Q_{33}=&-\gamma_{33}+\frac{1}{4}S_{31}\pi\rho a_1^2(a_2^2B_{1233}-a_1^2B_{1133})+\frac{1}{4}S_{32}\pi\rho a_2^2(5a_3^2B_{2333}-a_2^2B_{2233})\\
&+S_{33}\pi\rho\Big[-\frac{2P_0}{3\pi\rho^2 a_3^2}+\frac{5}{3}a_3^6A_{3333}-a_3^4a_1^2A_{1333}-\frac{5}{4}a_3^4B_{3333}+\frac{1}{4}a_1^2a_3^2B_{1333}\Big],
\end{aligned}
\tag{4.26}
$$

$$R_0 = -\kappa_0 + \frac{R}{2}\delta\Omega^2 + S_0\Big[\pi\rho\Big(-\frac{P_0}{\pi\rho^2 a_1^2} + A_1\Big) + \frac{1}{2}\Omega_{\rm DR}^2\Big]$$
$$+S_{21}\Big[\pi\rho a_1^2\Big(a_3^2 A_{13} - a_1^2 A_{11} + \frac{1}{2}B_{11}\Big) + \frac{I_{11}}{R^3} + O(R^{-5})\Big]$$
$$+S_{22}\Big[\pi\rho a_1^2(a_3^2 A_{13} - a_2^2 A_{12}) + O(R^{-5})\Big]$$
$$+O(R^{-4}) \times (S_{11},\ S_{12},\ S_{31},\ S_{32},\ S_{33}), \tag{4.27}$$
$$R_1 = -\kappa_1 + S_{21}\Big[\pi\rho\Big(-\frac{P_0}{\pi\rho^2 a_1^2} - a_1^2 a_3^2 A_{113} + 2a_1^4 A_{111} - \frac{1}{2}a_1^2 B_{111}\Big) + O(R^{-5})\Big]$$
$$+S_{22}\pi\rho a_1^2(a_2^2 A_{112} - a_3^2 A_{113}) + O(R^{-5}) \times S_0, \tag{4.28}$$
$$R_2 = -\kappa_2 + S_{21}\Big[\pi\rho a_1^2\Big(a_1^2 A_{112} - a_3^2 A_{123} - \frac{1}{2}B_{112}\Big) + O(R^{-5})\Big]$$
$$+S_{22}\pi\rho\Big(-\frac{2P_0}{\pi\rho^2 a_2^2} + 3a_1^2 a_2^2 A_{122} - a_1^2 a_3^2 A_{123}\Big) + O(R^{-5}) \times S_0, \tag{4.29}$$
$$R_3 = -\kappa_3 + S_{21}\Big[\pi\rho\Big(\frac{2P_0}{\pi\rho^2 a_3^2} + a_1^4 A_{113} - 3a_1^2 a_3^2 A_{133} - \frac{1}{2}a_1^2 B_{113}\Big) + O(R^{-5})\Big]$$
$$+S_{22}\pi\rho\Big(\frac{2P_0}{\pi\rho^2 a_3^2} + a_1^2 a_2^2 A_{123} - 3a_1^2 a_3^2 A_{133}\Big) + O(R^{-5}) \times S_0. \tag{4.30}$$

Since $(P/\rho)_S$ must vanish on the stellar surface, we have eight conditions,

$$a_1^4 Q_{11} + a_2^4 Q_{22} - a_1^2 a_2^2 Q_{12} = 0, \tag{4.31}$$
$$a_2^4 Q_{22} + a_3^4 Q_{33} - a_2^2 a_3^2 Q_{23} = 0, \tag{4.32}$$
$$a_3^4 Q_{33} + a_1^4 Q_{11} - a_3^2 a_1^2 Q_{13} = 0, \tag{4.33}$$
$$a_1^4 Q_{11} - a_2^4 Q_{22} + a_1^2 Q_1 - a_2^2 Q_2 = 0, \tag{4.34}$$
$$a_3^4 Q_{33} - a_1^4 Q_{11} + a_3^2 Q_3 - a_1^2 Q_1 = 0, \tag{4.35}$$

and

$$R_0 + a_3^2 R_3 = 0, \tag{4.36}$$
$$a_1^2 R_1 - a_3^2 R_3 = 0, \tag{4.37}$$
$$a_2^2 R_2 - a_3^2 R_3 = 0. \tag{4.38}$$

We can see that the equations for determining the coefficients of the 1PN velocity potential and the deformation of the binary system are separated into two combinations. One of them is constructed from Eqs. (2.78), (4.7) – (4.9) and (4.31) – (4.35), and determines the coefficients p, q, r, s, S_{11}, S_{12}, S_{31}, S_{32}, and S_{33}. These coefficients are associated with the star itself, and the triplane symmetric part of the Lagrangian displacement vectors. The other is the combination composed of Eqs. (2.79), (4.10) – (4.12) and (4.36) – (4.38). These equations determine the coefficients e, f, g, h, S_0, S_{21}, and S_{22}. These coefficients are associated with the binary motion, and the π-rotation symmetric part of the Lagrangian displacement vectors around the x_3-axis.

In the case of an isolated star, [18] we have two identical equations for Eqs. (4.31) – (4.35) when we substitute $\delta\Omega^2$ into Eqs. (4.34) and (4.35). However, for the case of a binary system, the five equations (4.31) – (4.35) are all independent up to $O(R^{-5})$ even if we substitute $\delta\Omega^2$ into Eqs. (4.34) and (4.35).

Contrastingly, when we substitute $\delta\Omega^2$ into Eq. (4.36), it is found that S_0 vanishes. This implies that only six of seven conditions (2.79), (4.10) – (4.12) and (4.36) – (4.38) are available. Here, we choose equations excluding Eq. (4.36).

There remains the problem of determining the remaining two coefficients e and S_0 because we have only one equation (4.10) (or (4.11) or (4.12)) for these two variables. There is one degree of freedom. This fact follows from point made by Chandrasekhar [26,27] and Bardeen [33] for an isolated star that there is no unique definition of the 1PN solution as the counterpart of a Newtonian binary solution.

Since the condition for connecting a 1PN solution to a Newtonian solution can be arbitrarily chosen, in this paper, we simply give the condition as

$$S_0 = 0, \tag{4.39}$$

i.e., we fix the 1PN correction to the orbital separation of the binary system. Then, we have two equations, (4.10) and (4.39), for solving for e and S_0.

Note that when we calculate Eqs. (4.34) and (4.35), we neglect terms such as

$$\frac{\Omega_{\rm DR}}{R} F_a \times (e,\ f,\ g,\ h) \tag{4.40}$$

in order to separate variables into two groups. We take terms up to $O(R^{-3})$ in Eqs. (4.34) and (4.35). In this calculation, we only count the order of $1/R$ and ignore the order of the *implicit* terms, such as F_a which is $O(R^{-3})$. However, we regard the term (4.40) as $O(R^{-4})$. Even if we include this term, the physical values, i.e., the energy, the angular momentum and the angular velocity, do not change.[§] Note that when we include the term (4.40), only the coefficients p, S_{11} and S_{12} change. This is because we can determine q, r, s, S_{31}, S_{32}, and S_{33} independent of p, S_{11} and S_{12} from Eqs. (2.78), (4.13), (4.14) and (4.31) – (4.33).

V. THE TOTAL ENERGY AND ANGULAR MOMENTUM

We take the total energy up to $O(R^{-3})$ at Newtonian order and up to $O(R^{-4})$ at 1PN order. Also the total angular momentum up to $O(1)$ at Newtonian order and up to $O(R^{-1})$ at 1PN order, except for $\Omega_{\rm DR}$.

A. Total energy

The total energy of the binary system is defined as [25]

$$\begin{aligned}
E &= 2\int_V d^3x\rho\Big[\frac{v^2}{2} - \frac{1}{2}U + \frac{1}{c^2}\Big(\frac{5}{8}v^4 + \frac{5}{2}v^2U + \frac{1}{2}\hat{\beta}_i v^i + \frac{P}{\rho}v^2 - \frac{5}{2}U^2\Big) + O(c^{-4})\Big],\\
&= E_{\rm N} + \frac{1}{c^2}E_{\rm PN} + O(c^{-4}),
\end{aligned} \tag{5.1}$$

where

$$\begin{aligned}
E_{\rm N} &= M_*\Big[\frac{1}{5}F_a^2\Omega_{\rm DR}^2(a_{1*}^2 + a_{2*}^2) + \frac{\tilde{R}^2 a_{1*}^2}{4}\Omega_{\rm DR}^2 - \frac{4}{5}\pi\rho A_{0*} - \frac{M_*}{\tilde{R}a_{1*}} - \frac{3\rlap{\,-}I_{11*}}{\tilde{R}^3 a_{1*}^3}\Big]\\
&\quad + O(R^{-5}),
\end{aligned} \tag{5.2}$$

$$\begin{aligned}
E_{\rm PN} &= E_{\rm N\to PN} + E_{v^4} + E_{v^2U^{1\to 1}} + E_{v^2U^{2\to 1}} + E_{v^i\beta_i^{1\to 1}} + E_{v^i\beta_i^{2\to 1}}\\
&\quad + E_{v^2P/\rho} + E_{(U^{1\to 1})^2} + E_{(U^{1\to 1})(U^{2\to 1})} + E_{(U^{2\to 1})^2},
\end{aligned} \tag{5.3}$$

$$\begin{aligned}
E_{\rm N\to PN} &= -E_{\rm N}\Big[\frac{1}{6}F_a^2\Omega_{\rm DR}^2(a_{1*}^2 + a_{2*}^2) + 4\pi\rho A_{0*} + \frac{65M_*}{12R} + \frac{75\rlap{\,-}I_{11*}}{4R^3}\Big]\\
&\quad + F_a^2(I_{11*} + I_{22*})\delta\Omega^2 + F_a^2\Omega_{\rm DR}^2(\delta I_{11*} + \delta I_{22*}) + \frac{M_*R^2}{4}\delta\Omega^2\\
&\quad + F_a\Omega_{\rm DR}^2 R\delta I_{1*}\\
&\quad + 2F_a\Omega_{\rm DR}\Big\{p(I_{11*} + I_{22*}) + q(3I_{1122*} + I_{1111*})\\
&\qquad\qquad + r(I_{2222*} + 3I_{1122*}) + s(I_{2233*} + I_{1133*})\Big\}\\
&\quad + R\Omega_{\rm DR}(eM_* + fI_{11*} + 3gI_{22*} + hI_{33*}) + 2\delta W_*\\
&\quad - \delta\int d^3x\rho U^{2\to 1},
\end{aligned} \tag{5.4}$$

$$E_{v^4} = \frac{5}{4}\Omega_{\rm DR}^4 R^4\Big[\frac{M_*}{16} + \frac{1}{2R^2}F_a^2(3I_{11*} + I_{22*}) + O(R^{-4})\Big], \tag{5.5}$$

$$E_{v^2U^{1\to 1}} = M_*\pi\rho A_{0*}\Omega_{\rm DR}^2 R^2\Big[1 + \frac{16F_a^2}{21R^2}(a_{1*}^2 + a_{2*}^2) + O(R^{-5})\Big], \tag{5.6}$$

[§] The change of the physical values is less than 1%, even near the ISCO.

$$
\begin{aligned}
E_{v^2U^{2\to1}} &= 5\Omega_{\rm DR}^2\Big[\frac{M_*^2R}{4}+\frac{M_*}{R}\Big\{F_a^2(I_{11*}+I_{22*})-F_aI_{11*}+\frac{3}{4}\not{I}_{11*}\Big\} \\
&\qquad +O(R^{-3})\Big],
\end{aligned} \tag{5.7}
$$

$$
\begin{aligned}
E_{v^i\beta_i^{1\to1}} &= -\frac{2}{35}M_*\pi\rho\Omega_{\rm DR}^2F_a^2a_{1*}^2a_{2*}^2\Big[\Big(\frac{7a_{1*}^2}{a_{2*}^2}-1\Big)A_{1*}+\Big(\frac{7a_{2*}^2}{a_{1*}^2}-1\Big)A_{2*} \\
&\qquad\qquad\qquad +2(a_{1*}^2+a_{2*}^2)A_{12*}\Big] \\
&\quad -\frac{R^2}{10}M_*\pi\rho\Omega_{\rm DR}^2(7A_{0*}+a_{2*}^2A_{2*}),
\end{aligned} \tag{5.8}
$$

$$
\begin{aligned}
E_{v^i\beta_i^{2\to1}} &= \frac{\Omega_{\rm DR}^2}{2}\Big[\frac{7M_*^2R}{4}+\frac{M_*}{2R}\Big(-14F_aI_{11*}+2F_aI_{22*}+I_{22*}+\frac{21}{2}\not{I}_{11*}\Big) \\
&\qquad +O(R^{-3})\Big],
\end{aligned} \tag{5.9}
$$

$$
E_{v^2P/\rho} = \frac{P_0}{5\rho}M_*R^2\Omega_{\rm DR}^2\Big[1+\frac{4F_a^2}{7R^2}(a_{1*}^2+a_{2*}^2)\Big], \tag{5.10}
$$

$$
E_{(U^{1\to1})^2} = -\frac{2}{7}(\pi\rho)^2M_*(11A_{0*}^2+a_{1*}^4A_{1*}^2+a_{2*}^4A_{2*}^2+a_{3*}^4A_{3*}^2), \tag{5.11}
$$

$$
E_{(U^{1\to1})(U^{2\to1})} = -\frac{8M_*^2}{R}\pi\rho A_{0*}\Big[1+\frac{41\not{I}_{11*}}{14R^2M_*}+O(R^{-5})\Big], \tag{5.12}
$$

$$
E_{(U^{2\to1})^2} = -5\Big[\frac{M_*^3}{R^2}+\frac{M_*^2}{R^4}(6\not{I}_{11*}+I_{11*})+O(R^{-6})\Big]. \tag{5.13}
$$

In the above equations, we have used the conserved mass

$$
\begin{aligned}
M_* &\equiv \int d^3x\rho\Big[1+\frac{1}{c^2}\Big(\frac{v^2}{2}+3U\Big)\Big], \\
&= M\Big[1+\frac{1}{c^2}\Big\{\frac{\Omega_{\rm DR}^2}{2}\Big(\frac{F_a^2}{5}(a_1^2+a_2^2)+\frac{R^2}{4}\Big)+\frac{12\pi\rho}{5}A_0+\frac{3M}{R}+\frac{9\not{I}_{11}}{R^3}\Big\}\Big],
\end{aligned} \tag{5.14}
$$

instead of the Newtonian mass M, and we have defined $\tilde{R} \equiv R/a_1$. Also we substitute the mean radius of the star defined by the conserved mass as

$$
\begin{aligned}
a_* &\equiv \Big(\frac{M_*}{4\pi\rho/3}\Big)^{1/3} \\
&= a_1(\alpha_2\alpha_3)^{1/3}\Big[1+\frac{1}{c^2}\Big\{\frac{\Omega_{\rm DR}^2}{6}\Big(\frac{F_a^2}{5}(a_1^2+a_2^2)+\frac{R^2}{4}\Big)+\frac{4\pi\rho}{5}A_0 \\
&\qquad\qquad +\frac{M}{R}+\frac{3\not{I}_{11}}{R^3}\Big\}\Big],
\end{aligned} \tag{5.15}
$$

and a_{i*} defined by $a_{1*}a_{2*}a_{3*}=a_*^3$, $a_{2*}/a_{1*}=a_2/a_1$ and $a_{3*}/a_{1*}=a_3/a_1$. In Eq. (5.4), δI_1, δI_{ii}, and δW denote the perturbed dipole moment, the perturbed quadrupole moment, and the perturbed self-gravity potential energy, respectively. These terms are given in Chandrasekhar's textbook [29] and in our previous paper. [18] We represent these terms in Appendix B.

The last term on the right-hand side of Eq. (5.4) is calculated from

$$
\delta\int d^3x\rho U^{2\to1} = \int d^3x\rho\delta U^{2\to1}+\int d^3x\rho\sum_l\xi_l\frac{\partial U^{2\to1}}{\partial x_l}, \tag{5.16}
$$

where the first term on the right-hand side of the above equation denotes the gravitational potential energy which the nondisplaced element of star 1 experiences due to the displaced potential of star 2. On the other hand, the second term denotes the gravitational potential energy which the displaced element of star 1 experiences due to the nondisplaced potential of star 2. For the Lagrangian displacement vectors given in Eqs. (3.4) – (3.11), the first and second terms of Eq. (5.16) are equal, and calculated as

$$
\begin{aligned}
\delta \int d^3x\rho U^{2\to 1} &= 2\int d^3x\rho\delta U^{2\to 1}, \\
&= 2\int d^3x\rho \sum_l \xi_l \frac{\partial U^{2\to 1}}{\partial x_l}, \\
&= \frac{2}{c^2}\Big[S_{11}\Big\{\frac{M}{R^3}(2I_{11}+I_{33})+O(R^{-5})\Big\} \\
&\qquad +S_{12}\Big\{\frac{M}{R^3}(I_{33}-I_{22})+O(R^{-5})\Big\} \\
&\qquad +S_{31}\Big\{\frac{M}{R^3}\Big(\frac{2}{3}I_{1111}+I_{1122}\Big)+O(R^{-5})\Big\} \\
&\qquad +S_{32}\Big\{\frac{M}{R^3}\Big(I_{2233}-\frac{1}{3}I_{2222}\Big)+O(R^{-5})\Big\} \\
&\qquad -S_{33}\Big\{\frac{M}{R^3}\Big(2I_{1133}+\frac{1}{3}I_{3333}\Big)+O(R^{-5})\Big\} \\
&\qquad -\frac{1}{2}S_0\Big\{\frac{M^2}{R^2}+\frac{9M\not{I}_{11}}{R^4}+O(R^{-6})\Big\} \\
&\qquad -S_{21}\Big\{\frac{MI_{11}}{2R^2}+\frac{9I_{11}}{4R^4}\not{I}_{11}+\frac{9M}{4R^4}\not{I}_{1111}+\frac{3M}{R^4}I_{1133}+O(R^{-6})\Big\} \\
&\qquad -S_{22}\Big\{\frac{3M}{R^4}(I_{1133}-I_{1122})+O(R^{-6})\Big\}\Big].
\end{aligned} \tag{5.17}
$$

B. Total angular momentum

The total angular momentum is written as [25]

$$
\begin{aligned}
J &= 2\int_V d^3x\rho\Big[v_\varphi\Big\{1+\frac{1}{c^2}\Big(v^2+6U+\frac{P}{\rho}\Big)\Big\}+\frac{\hat{\beta}_\varphi}{c^2}+O(c^{-4})\Big], \\
&= J_{\rm N}+\frac{1}{c^2}J_{\rm PN}+O(c^{-4}),
\end{aligned} \tag{5.18}
$$

where

$$
v_\varphi = -x_2v_1+\Big(x_1+\frac{R}{2}\Big)v_2, \tag{5.19}
$$

$$
\hat{\beta}_\varphi = -x_2\hat{\beta}_1+\Big(x_1+\frac{R}{2}\Big)\hat{\beta}_2, \tag{5.20}
$$

$$
J_{\rm N} = 2\Omega_{\rm DR}M_*\Big[\frac{F_a}{5}(a_{1*}^2-a_{2*}^2)+\frac{\tilde{R}^2}{4}a_{1*}^2\Big], \tag{5.21}
$$

$$
\begin{aligned}
J_{\rm PN} &= J_{\rm N\to PN}+J_{v_\varphi v^2}+J_{v_\varphi U^{1\to 1}}+J_{v_\varphi U^{2\to 1}}+J_{v_\varphi P/\rho} \\
&\quad +J_{\beta_\varphi^{1\to 1}}+J_{\beta_\varphi^{2\to 1}},
\end{aligned} \tag{5.22}
$$

$$
\begin{aligned}
J_{\rm N\to PN} &= -J_{\rm N}\Big[\frac{1}{6}F_a^2\Omega_{\rm DR}^2(a_{1*}^2+a_{2*}^2)+4\pi\rho A_{0*}+\frac{65M_*}{12R}+\frac{75\not{I}_{11*}}{4R^3}\Big] \\
&\quad +2\Big[F_a(I_{11*}-I_{22*})\delta\Omega+F_a\Omega_{\rm DR}(\delta I_{11*}-\delta I_{22*})+\frac{M_*R^2}{4}\delta\Omega \\
&\qquad +\frac{R}{2}(F_a+1)\Omega_{\rm DR}\delta I_{1*}+p(I_{11*}-I_{22*}) \\
&\qquad +q(I_{1111*}-3I_{1122*})+r(3I_{1122*}-I_{2222*})+s(I_{1133*}-I_{2233*}) \\
&\qquad +\frac{R}{2}(eM_*+fI_{11*}+3gI_{22*}+hI_{33*})\Big],
\end{aligned} \tag{5.23}
$$

$$
\begin{aligned}
J_{v_\varphi v^2} &= \frac{R^4}{2}\Omega_{\rm DR}^3\Big[\frac{M_*}{4}+\frac{F_a}{R^2}\Big\{3(F_a+1)I_{11*}+(F_a-1)I_{22*}\Big\} \\
&\qquad +O(R^{-4})\Big],
\end{aligned} \tag{5.24}
$$

$$J_{v_\varphi U^{1\to 1}} = \frac{12}{5} M_* \pi \rho \Omega_{\rm DR} R^2 A_{0*} \Big[1 + \frac{16 F_a}{21 R^2} (a_{1*}^2 - a_{2*}^2) + O(R^{-5}) \Big], \tag{5.25}$$

$$J_{v_\varphi U^{2\to 1}} = 3\Omega_{\rm DR} \Big[M_*^2 R + \frac{M_*}{R} \Big\{ 2(F_a - 1) I_{11*} - 4 F_a I_{22*} + 3 \not{I}_{11*} \Big\} + O(R^{-3}) \Big], \tag{5.26}$$

$$J_{v_\varphi P/\rho} = \frac{P_0}{5\rho} \Omega_{\rm DR} M_* R^2 \Big[1 + \frac{4 F_a}{7 R^2} (a_{1*}^2 - a_{2*}^2) \Big], \tag{5.27}$$

$$J_{\beta_\varphi^{1\to 1}} = -\frac{32}{105} M_* \pi \rho \Omega_{\rm DR} F_a \Big[A_{0*} (a_{1*}^2 - a_{2*}^2) + O(R^{-3}) \Big] - \frac{R^2}{5} M_* \pi \rho \Omega_{\rm DR} (7 A_{0*} + a_{2*}^2 A_{2*}), \tag{5.28}$$

$$J_{\beta_\varphi^{2\to 1}} = \Omega_{\rm DR} \Big[\frac{7 M_*^2 R}{4} + \frac{M_*}{2R} \Big\{ -7(F_a + 1) I_{11*} + F_a I_{22*} + \frac{21 \not{I}_{11*}}{2} \Big\} + O(R^{-3}) \Big], \tag{5.29}$$

and

$$\delta\Omega = \frac{\delta\Omega^2}{2\Omega_{\rm DR}}. \tag{5.30}$$

VI. NUMERICAL RESULTS

In the following subsections, we calculate the equilibrium sequence of the irrotational binary system.

A. Normalization

First of all, we introduce non-dimensional parameters according to

$$\tilde{p} \equiv \frac{p}{(M_*^3/a_*^5)^{1/2}}, \quad \tilde{q} \equiv \frac{q}{(M_*/a_*^3)^{3/2}}, \quad \tilde{r} \equiv \frac{r}{(M_*/a_*^3)^{3/2}}, \quad \tilde{s} \equiv \frac{s}{(M_*/a_*^3)^{3/2}}, \tag{6.1}$$

$$\tilde{e} \equiv \frac{e}{(M_*/a_*)^{3/2}}, \quad \tilde{f} \equiv \frac{f}{(M_*^3/a_*^7)^{1/2}}, \quad \tilde{g} \equiv \frac{g}{(M_*^3/a_*^7)^{1/2}}, \quad \tilde{h} \equiv \frac{h}{(M_*^3/a_*^7)^{1/2}}, \tag{6.2}$$

$$\tilde{S}_{11} \equiv \frac{S_{11}}{M_*/a_*}, \quad \tilde{S}_{12} \equiv \frac{S_{12}}{M_*/a_*},$$
$$\tilde{S}_{31} \equiv \frac{S_{31}}{M_*/a_*^3}, \quad \tilde{S}_{32} \equiv \frac{S_{32}}{M_*/a_*^3}, \quad \tilde{S}_{33} \equiv \frac{S_{33}}{M_*/a_*^3}, \tag{6.3}$$

$$\tilde{S}_0 \equiv \frac{S_0}{M_*}, \quad \tilde{S}_{21} \equiv \frac{S_{21}}{M_*/a_*^2}, \quad \tilde{S}_{22} \equiv \frac{S_{22}}{M_*/a_*^2}, \tag{6.4}$$

$$\tilde{\Omega}_{\rm DR}^2 \equiv \frac{\Omega_{\rm DR}^2}{M_*/a_*^3}, \quad \delta\tilde{\Omega}^2 \equiv \frac{\delta\Omega^2}{M_*^2/a_*^4}, \quad \delta\tilde{\Omega} \equiv \frac{\delta\Omega}{(M_*^3/a_*^5)^{1/2}}, \quad \tilde{P}_0 \equiv \frac{P_0}{\rho M_*/a_*}. \tag{6.5}$$

Using these parameters with the condition $S_0 = 0$, we can rewrite two groups of equations into non-dimensional forms. One of these is (2.78), (4.7) – (4.9), (4.31) – (4.35), and the other is (2.79), (4.10) – (4.12), (4.37) and (4.38). From these fifteen equations, we can determine fifteen variables, $\tilde{p}$, $\tilde{q}$, $\tilde{r}$, $\tilde{s}$, $\tilde{S}_{11}$, $\tilde{S}_{12}$, $\tilde{S}_{31}$, $\tilde{S}_{32}$ and $\tilde{S}_{33}$, and $\tilde{e}$, $\tilde{f}$, $\tilde{g}$, $\tilde{h}$, $\tilde{S}_{21}$ and $\tilde{S}_{22}$.

After determination of the variables, we can calculate the total energy and total angular momentum of the binary system from Eqs. (5.1) and (5.18). In the numerical calculations, these values must be normalized. The normalized energy and angular momentum are written as

$$\tilde{E} \equiv \frac{E}{M_*^2/a_*} = \tilde{E}_{\rm N} + C_{\rm s} \tilde{E}_{\rm PN}, \tag{6.6}$$

$$\tilde{J} \equiv \frac{J}{(M_*^3 a_*)^{1/2}} = \tilde{J}_{\rm N} + C_{\rm s} \tilde{J}_{\rm PN}, \tag{6.7}$$

where $C_{\rm s}$ is the compactness parameter defined by

$$C_{\rm s} \equiv \frac{M_*}{c^2 a_*}. \tag{6.8}$$

We consider the case $C_{\rm s} \ll 1$ because of the 1PN approximation.

At 1PN order, the center of mass of the binary system deviates from the value defined at Newtonian order. The center of mass at 1PN order is defined by using the conserved mass as

$$x_*^i \equiv \frac{1}{M_*}\int d^3x \rho_* x^i. \tag{6.9}$$

The x_1 component is

$$x_*^1 = \frac{1}{M_*}\Big[\delta I_1 + \frac{1}{c^2}\Big\{\Big(\frac{\Omega_{\rm DR}^2}{2}F_a R - \frac{3M}{R^2}\Big)I_{11} + O(R^{-4})\Big\}\Big], \tag{6.10}$$

Then, the orbital separation of the 1PN order is written as

$$R_* = R\Big[1 + \frac{1}{c^2}\Big\{\frac{a_1^2}{5}\Big(\Omega_{\rm DR}^2 F_a - \frac{6M}{R^3}\Big) + \frac{1}{R}\Big(S_0 + \frac{1}{5}S_{21}a_1^2\Big)\Big\}\Big]. \tag{6.11}$$

Once an equilibrium sequence is obtained, we search for the minimum point of the total energy. If we find it, we call it the ISCO.

B. Ellipsoidal approximation

The ellipsoidal approximation, in which the equilibrium configuration is assumed to have an ellipsoidal figure, is useful when we study features of stars or binary systems in the 1PN approximation, [17,19,34] because the whole calculation is carried out by setting $S_{ij} = 0$, and hence is greatly simplified. The ellipsoidal approximation gives an exact solution for a rotating incompressible star in the Newtonian theory. However, it gives only an approximate solution for the 1PN case. If the ellipsoidal approximation is truly a robust approximation, it becomes a useful method for the study of the 1PN effects. This is the motivation for our investigation of the validity of the ellipsoidal approximation.

In the ellipsoidal approximation, a solution is obtained by setting $S_{ij} = 0$ in all the equations. After we set $S_{ij} = 0$, we can calculate the velocity potential at 1PN order from Eqs. (2.78) and (4.7) – (4.9), and also (2.79) and (4.10) – (4.12). We note that in the ellipsoidal approximation, the boundary condition (4.5) is not satisfied.

C. Results

The results are shown in Figs. 2 – 5. In Fig. 2, we represent the total energy and total angular momentum as functions of the normalized separation of the binary system R_*/a_* and the normalized orbital angular velocity $\Omega/\sqrt{M_*/a_*^3}$. We can see from this figure that the minimum point of the total angular momentum is slightly different from that of the total energy. The deviation between the location of the minimums of the energy and angular momentum comes from the fact that we assume a_*/R_* is a small parameter and expand the energy up to $O(R_*^{-4})$ and the angular momentum up to $O(R_*^{-1})$. In any case, we may expect that the ISCO is located near the two minimums. It is found that even if we increase the compactness parameter $C_{\rm s}$, the angular velocity at the ISCO in units of $\sqrt{M_*/a_*^3}$ has almost the same value as in the Newtonian case, while the orbital separation decreases. This feature of the angular velocity at the ISCO is different from that in the case of the corotational binary system. In such a case, we have obtained the result that when we increase the compactness parameter, the angular velocity at the ISCO increases. [19] In a previous paper investigating irrotational and incompressible stars, [18] we found the instability driven by the deformation of 1PN order. We regard such an instability point as the bifurcation point to a new sequence. However, in the present study, we do not find such an instability point throughout the equilibrium sequence of the binary system until the ISCO.

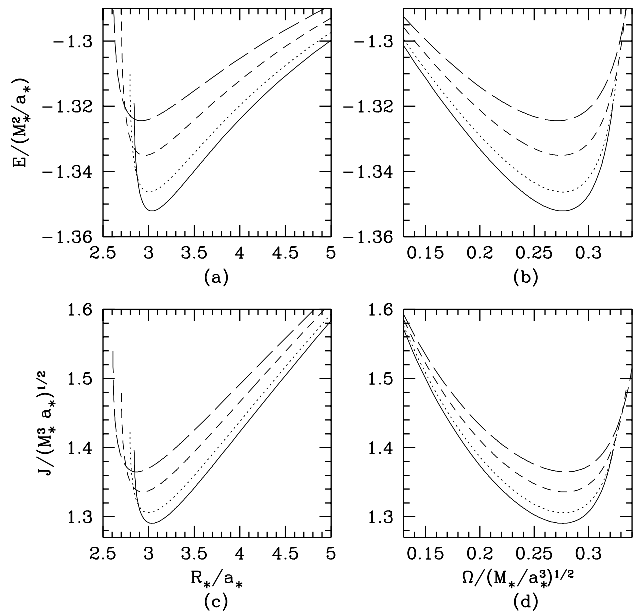

FIG. 2. The total energy and the total angular momentum of the *full* calculation as functions of the orbital separation R_*/a_* and the angular velocity $\Omega/(M_*/a_*^3)^{1/2}$. The solid, dotted, dashed, and long-dashed lines denote the cases of $C_{\rm s}=0$ (Newtonian), 0.01, 0.03, and 0.05, respectively.

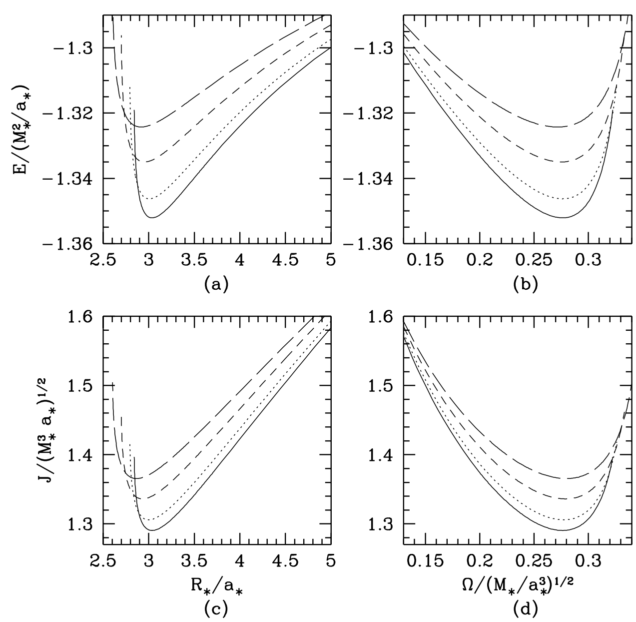

FIG. 3. The total energy and the total angular momentum of the *ellipsoidal* approximation as functions of the orbital separation R_*/a_* and the angular velocity $\Omega/(M_*/a_*^3)^{1/2}$. The identifications of the lines are the same as in Fig. 2.

In Fig. 3, the total energy and total angular momentum in the ellipsoidal approximation are shown as functions of the normalized separation of the binary system and the normalized orbital angular velocity. It is found that the quantitative features of the energy and angular momentum are the same as in Fig. 2. Also, we find that values at 1PN order ($\delta\Omega_{\rm PNe}$, $E_{\rm PNe}$ and $J_{\rm PNe}$) differ by only a few percent outside the ISCO from those of the full calculation, even

if the figure is assumed to be an ellipsoid. This implies that the ellipsoidal approximation gives good approximate results.

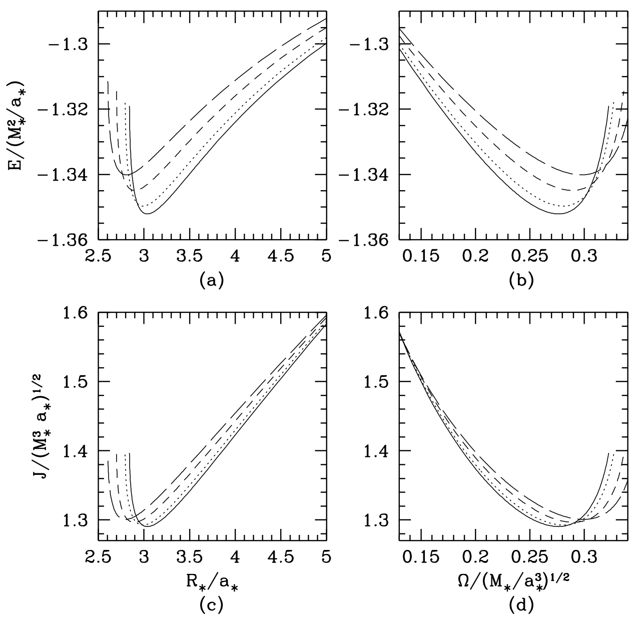

FIG. 4. The total energy and the total angular momentum of the approximation in which we neglect the velocity potential at 1PN order as functions of the orbital separation R_*/a_* and the angular velocity $\Omega/(M_*/a_*^3)^{1/2}$. The identifications of the lines are the same as in Fig. 2.

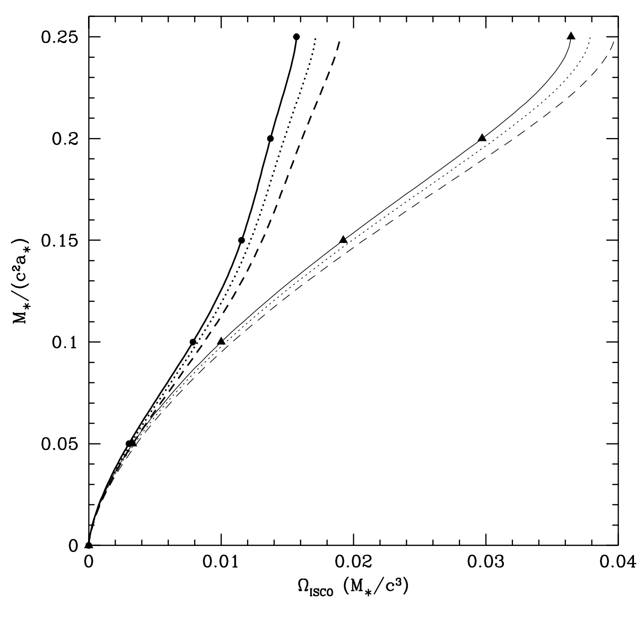

FIG. 5. The relation between the angular velocity at the ISCO $\Omega_{\rm ISCO}(M_*/c^3)$ and the compactness parameter $C_{\rm s} = M_*/c^2a_*$. The thick and thin lines denote the cases of the full calculation and the absence of the 1PN velocity potential, respectively. Filled circles and triangles denote the results of our calculation. Solid, dotted and dashed lines are the fitting lines for $n = 0$ (incompressible case), 0.5 and 0.7, where n denotes the polytropic index.

However, neglect of the velocity potential of 1PN order has an effect on the quantitative nature of the energy and angular momentum. We can see from Fig. 4 that the angular velocity at the minimum point of the energy and angular momentum increases when we increase the compactness parameter. Figure 4 is obtained by setting the velocity potential at 1PN order to zero and solving only Eqs. (4.7) – (4.9) and (4.31) – (4.35), and also (4.10) – (4.12), (4.37) and (4.38).

In Fig. 5, we show the relation between the compactness parameter $C_{\rm s} = M_*/c^2 a_*$ and the angular velocity at the ISCO in units of c^3/M_*. Although the 1PN approximation is not valid, we take $C_{\rm s}$ up to 0.25, which we regard as the realistic compactness parameter of a neutron star, in order to compare our results with those of Lombardi, Rasio and Shapiro [17] (see Fig. 4 in their paper). In Fig. 5, the thick and thin solid lines denote the cases of the full calculation and the absence of the 1PN velocity potential. These lines come from our calculation for an incompressible fluid. The dotted and dashed lines denote the "compressible" cases, whose polytropic indices are 0.5 and 0.7, respectively. We plot these lines by assuming that the 1PN correction to the angular velocity is the same value as in the incompressible case if the stars in the binary system composed of compressible fluid have a polytropic index less than 1, i.e., the hard equation of state. Also, we adopt the angular velocity at the ISCO given by Uryū and Eriguchi [12] as that of the irrotational and compressible binary system in the Newtonian theory.

We can see from Fig. 5 that the angular velocity at the ISCO is overestimated when we neglect the 1PN velocity potential. Figure 5 agrees qualitatively with that of Lombardi, Rasio and Shapiro. [17] However, our results suggest that the angular velocities given by Lombardi, Rasio and Shapiro are overestimated because of the absence of the 1PN velocity potential in their calculation. The reason that the total force of the system balances with the smaller orbital angular velocity than that in the case of the absence of the 1PN velocity potential is as follows. We regard the energy minimum point as the ISCO in this paper. As the stars in the binary system become closer and closer to each other, the energy terms with a minus sign, such as the binding energy term, decrease and those with a plus sign, such as the rotation energy term, increase. At the ISCO, the increase of the energy terms with a plus sign is equal to the decrease of those with a minus sign. When the star possesses internal motion, such motion can produce a part of the energy with a plus sign. Therefore, if the star with the 1PN velocity potential has the same ellipsoidal figure as that without the 1PN velocity potential, the energy of the orbital motion, that is to say, the orbital angular velocity, can decrease.

Finally, we summarize the results in Tables I – VI. In Table I, the angular velocity, energy and angular momentum of the Newtonian and 1PN orders are shown along the sequence of the binary system. In this table, the symbol † denotes the location of the ISCO of the Newtonian binary system, i.e., the point at which the total energy has its minimum.

In Table II, the total angular velocity, total energy, and total angular momentum at 1PN order are shown. The symbol † denotes the location of the ISCO. In Table III, we give the coefficients of the velocity potential and deformation associated with the star itself, and in Table IV those associated with the binary motion are represented. Also, we show the results of the ellipsoidal approximation in Tables V and VI. We can see from Table V that the values of the energy, angular momentum and angular velocity at 1PN order in the ellipsoidal approximation deviate by only a few percent from those in the full calculation (Table I).

TABLE I. Equilibrium sequences of the irrotational Darwin-Riemann ellipsoids at Newtonian order and the angular velocity, the energy, and the angular momentum at 1PN order along the sequences. The symbol † denotes the energy minimum point of the Newtonian sequence, i.e., the ISCO defined in this paper.

R/a_1	R_*/a_*	a_2/a_1	a_3/a_1	$\tilde{\Omega}_{\rm DR}$	$\tilde{E}_{\rm N}$	$\tilde{J}_{\rm N}$	$\delta\tilde{\Omega}$	$\tilde{E}_{\rm PN}$	$\tilde{J}_{\rm PN}$
6.00	6.071	0.9824	0.9828	9.461(-2)	-1.282	1.743	0.1249	2.944(-2)	-0.7523
5.50	5.584	0.9771	0.9777	0.1073	-1.289	1.673	0.1401	5.964(-2)	-0.4832
5.00	5.103	0.9693	0.9705	0.1229	-1.298	1.600	0.1584	9.933(-2)	-0.1947
4.50	4.628	0.9576	0.9597	0.1424	-1.308	1.525	0.1806	0.1523	0.1157
4.00	4.165	0.9391	0.9430	0.1671	-1.319	1.450	0.2079	0.2239	0.4498
3.50	3.722	0.9078	0.9160	0.1986	-1.333	1.377	0.2412	0.3212	0.8081
3.00	3.317	0.8514	0.8692	0.2382	-1.346	1.315	0.2802	0.4511	1.190
†2.582	3.037	0.7671	0.8013	0.2763	-1.352	1.291	0.3147	0.5879	1.548
2.50	2.992	0.7446	0.7831	0.2839	-1.352	1.292	0.3215	0.6225	1.641
2.00	2.842	0.5586	0.6238	0.3226	-1.319	1.397	0.3984	0.9807	2.856

TABLE II. Equilibrium sequences of the irrotational Darwin-Riemann ellipsoids at 1PN order with the compactness parameters $M_*/c^2a_* = 0.01$, 0.03, and 0.05. The symbol † denotes the point of the ISCO.

R/a_1	R_*/a_*	a_2/a_1	a_3/a_1	$\tilde{\Omega}$	$\tilde{E}$	$\tilde{J}$
			$M_*/c^2a_* = 0.01$			
6.00	5.987	0.9824	0.9828	9.585(-2)	-1.282	1.736
5.50	5.506	0.9771	0.9777	0.1087	-1.289	1.668
5.00	5.030	0.9693	0.9705	0.1244	-1.297	1.598
4.50	4.561	0.9576	0.9597	0.1442	-1.306	1.526
4.00	4.104	0.9391	0.9430	0.1691	-1.317	1.454
3.50	3.666	0.9078	0.9160	0.2010	-1.329	1.385
3.00	3.265	0.8514	0.8692	0.2410	-1.341	1.327
†2.618	3.008	0.7762	0.8086	0.2761	-1.346	1.306
2.50	2.944	0.7446	0.7831	0.2871	-1.346	1.308
2.00	2.796	0.5586	0.6238	0.3266	-1.309	1.426
			$M_*/c^2a_* = 0.03$			
6.00	5.819	0.9824	0.9828	9.835(-2)	-1.281	1.721
5.50	5.350	0.9771	0.9777	0.1115	-1.288	1.658
5.00	4.885	0.9693	0.9705	0.1276	-1.295	1.594
4.50	4.428	0.9576	0.9597	0.1478	-1.303	1.529
4.00	3.981	0.9391	0.9430	0.1733	-1.313	1.463
3.50	3.553	0.9078	0.9160	0.2058	-1.323	1.401
3.00	3.161	0.8514	0.8692	0.2466	-1.332	1.351
†2.698	2.958	0.7951	0.8238	0.2747	-1.335	1.336
2.50	2.848	0.7446	0.7831	0.2936	-1.333	1.341
2.00	2.703	0.5586	0.6238	0.3346	-1.290	1.483
			$M_*/c^2a_* = 0.05$			
6.00	5.651	0.9824	0.9828	0.1009	-1.281	1.706
5.50	5.193	0.9771	0.9777	0.1143	-1.286	1.648
5.00	4.740	0.9693	0.9705	0.1308	-1.293	1.590
4.50	4.294	0.9576	0.9597	0.1514	-1.300	1.531
4.00	3.858	0.9391	0.9430	0.1775	-1.308	1.472
3.50	3.440	0.9078	0.9160	0.2106	-1.316	1.417
3.00	3.057	0.8514	0.8692	0.2522	-1.323	1.374
†2.795	2.919	0.8156	0.8403	0.2715	-1.324	1.366
2.50	2.751	0.7446	0.7831	0.3000	-1.321	1.373
2.00	2.610	0.5586	0.6238	0.3425	-1.270	1.540

TABLE III. Coefficients of the 1PN velocity potential and the Lagrangian displacement vectors associated with the star itself shown along the equilibrium sequence of the irrotational Darwin-Riemann ellipsoid.

R/a_1	$\tilde{p}$	$\tilde{q}$	$\tilde{r}$	$\tilde{s}$	$\tilde{S}_{11}$	$\tilde{S}_{12}$	$\tilde{S}_{31}$	$\tilde{S}_{32}$	$\tilde{S}_{33}$
6.00	-1.634(-2)	-6.281(-4)	5.725(-5)	4.333(-5)	3.205(-3)	-1.997(-2)	-1.950(-4)	-1.851(-5)	6.821(-5)
5.50	-1.937(-2)	-9.253(-4)	8.352(-5)	5.711(-5)	8.525(-4)	-1.978(-2)	-3.235(-4)	-3.191(-5)	1.152(-4)
5.00	-2.296(-2)	-1.412(-3)	1.256(-4)	7.251(-5)	-2.921(-3)	-1.884(-2)	-5.599(-4)	-5.826(-5)	2.046(-4)
4.50	-2.686(-2)	-2.248(-3)	1.948(-4)	8.096(-5)	-8.970(-3)	-1.665(-2)	-1.016(-3)	-1.141(-4)	3.862(-4)
4.00	-2.990(-2)	-3.760(-3)	3.104(-4)	4.680(-5)	-1.855(-2)	-1.258(-2)	-1.943(-3)	-2.436(-4)	7.863(-4)
3.50	-2.752(-2)	-6.660(-3)	4.937(-4)	-1.711(-4)	-3.264(-2)	-6.665(-3)	-3.896(-3)	-5.781(-4)	1.760(-3)
3.00	-1.683(-3)	-1.249(-2)	6.546(-4)	-1.149(-3)	-4.573(-2)	-5.496(-3)	-7.864(-3)	-1.548(-3)	4.448(-3)
2.50	0.1209	-2.332(-2)	-7.605(-4)	-5.280(-3)	1.111(-4)	-5.723(-2)	-1.216(-2)	-4.555(-3)	1.313(-2)
2.00	0.6365	-2.618(-2)	-2.067(-2)	-1.992(-2)	7.509(-1)	-5.522(-1)	2.515(-2)	-1.447(-2)	5.382(-2)

TABLE IV. Coefficients of the 1PN velocity potential and the Lagrangian displacement vectors associated with the binary motion shown along the equilibrium sequence of the irrotational Darwin-Riemann ellipsoid.

R/a_1	$\tilde{e}$	$\tilde{f}$	$\tilde{g}$	$\tilde{h}$	$\tilde{S}_{21}$	$\tilde{S}_{22}$
6.00	0.1750	-2.520(-2)	-3.005(-2)	-2.997(-2)	1.465(-3)	-1.608(-3)
5.50	0.2113	-2.517(-2)	-3.178(-2)	-3.164(-2)	2.264(-3)	-2.484(-3)
5.00	0.2567	-2.465(-2)	-3.394(-2)	-3.368(-2)	3.649(-3)	-3.999(-3)
4.50	0.3142	-2.320(-2)	-3.678(-2)	-3.627(-2)	6.185(-3)	-6.766(-3)
4.00	0.3880	-1.991(-2)	-4.077(-2)	-3.970(-2)	1.116(-2)	-1.216(-2)
3.50	0.4835	-1.278(-2)	-4.697(-2)	-4.450(-2)	2.175(-2)	-2.351(-2)
3.00	0.6057	3.204(-3)	-5.798(-2)	-5.167(-2)	4.663(-2)	-4.944(-2)
2.50	0.7498	4.035(-2)	-8.037(-2)	-6.307(-2)	0.1099	-0.1105
2.00	0.8687	0.1120	-0.1227	-8.151(-2)	0.2634	-0.2280

TABLE V. The 1PN angular velocity, energy, and angular momentum in the ellipsoidal approximation along the equilibrium sequence of the irrotational Darwin-Riemann ellipsoid. The subscript "e" denotes the case of the *ellipsoidal* approximation.

R/a_1	R_*/a_*	$\delta\tilde{\Omega}_{\rm e}$	$\tilde{E}_{\rm PNe}$	$\tilde{J}_{\rm PNe}$
6.00	6.071	0.1249	2.940(-2)	-0.7527
5.50	5.584	0.1401	5.961(-2)	-0.4835
5.00	5.103	0.1584	9.935(-2)	-0.1945
4.50	4.628	0.1808	0.1524	0.1167
4.00	4.165	0.2084	0.2244	0.4526
3.50	3.722	0.2426	0.3225	0.8149
3.00	3.317	0.2837	0.4541	1.204
2.50	2.992	0.3276	0.6165	1.626
2.00	2.842	0.3527	0.7600	2.153

TABLE VI. Coefficients of the 1PN velocity potential in the ellipsoidal approximation along the equilibrium sequence of the irrotational Darwin-Riemann ellipsoid. The subscript "e" denotes the case of the *ellipsoidal* approximation.

R/a_1	$\tilde{p}_{\rm e}$	$\tilde{q}_{\rm e}$	$\tilde{r}_{\rm e}$	$\tilde{s}_{\rm e}$	$\tilde{e}_{\rm e}$	$\tilde{f}_{\rm e}$	$\tilde{g}_{\rm e}$	$\tilde{h}_{\rm e}$
6.00	-1.853(-2)	-6.155(-4)	4.426(-5)	4.445(-5)	0.1750	-2.537(-2)	-3.000(-2)	-2.997(-2)
5.50	-2.157(-2)	-9.015(-4)	5.892(-5)	5.939(-5)	0.2113	-2.546(-2)	-3.168(-2)	-3.164(-2)
5.00	-2.489(-2)	-1.365(-3)	7.629(-5)	7.751(-5)	0.2567	-2.519(-2)	-3.376(-2)	-3.368(-2)
4.50	-2.791(-2)	-2.147(-3)	8.957(-5)	9.302(-5)	0.3141	-2.427(-2)	-3.642(-2)	-3.628(-2)
4.00	-2.881(-2)	-3.529(-3)	6.887(-5)	7.960(-5)	0.3879	-2.220(-2)	-4.001(-2)	-3.972(-2)
3.50	-2.218(-2)	-6.096(-3)	-1.043(-4)	-6.787(-5)	0.4833	-1.810(-2)	-4.518(-2)	-4.455(-2)
3.00	8.177(-3)	-1.107(-2)	-8.948(-4)	-7.645(-4)	0.6055	-1.052(-2)	-5.335(-2)	-5.185(-2)
2.50	0.1068	-2.057(-2)	-4.058(-3)	-3.635(-3)	0.7515	2.764(-3)	-6.759(-2)	-6.380(-2)
2.00	0.3336	-3.347(-2)	-1.531(-2)	-1.414(-2)	0.8843	2.684(-2)	-9.351(-2)	-8.377(-2)

VII. SUMMARY AND DISCUSSION

A. Summary

Using the scheme developed in a previous paper, [18] we have investigated the equilibrium sequences of irrotational and incompressible binary systems in the 1PN approximation. Our results presented in the previous section should be useful to check the numerical code for solving the irrotational binary problem. The conclusions are as follows.
(1) Due to the 1PN effect, the orbital separation at the ISCO decreases in proportion to the compactness parameter M_*/c^2a_*.
(2) The orbital angular velocity at the ISCO in units of $\sqrt{M_*/a_*^3}$ has almost the same value as in the Newtonian case, even if we increase the compactness parameter.
(3) The ellipsoidal approximation gives fairly accurate results throughout the equilibrium sequence.
(4) It is important to include the velocity potential at 1PN order.

B. Discussion

Finally, we discuss the velocity potential at 1PN order. As stated in a previous paper, [18] the internal velocity has an x_3 component. Therefore, we suggest that when we solve an irrotational binary problem numerically, we must include the x_3 component of the internal velocity.

Next, we discuss the effect of the deformation at Newtonian order. Even in the incompressible case, stars in the binary system deform from ellipsoidal figures at Newtonian order. [13] We can estimate the contribution from the octupole moment on the angular velocity. First, we assume that the deformation from an ellipsoidal figure is a small perturbation. Accordingly, the deformation is expressed by the Lagrangian displacement vectors (3.10) and (3.11) for the octupole moment. Here we neglect the contribution of the hexadecapole and other moments as higher order terms. Then, the angular velocity is written

$$\begin{aligned}\Omega^2 = &\frac{2M}{R^3} + \frac{18 \not{I}_{11}}{R^5} - \frac{2}{MR}\Big[\frac{2M}{R^3} + \frac{18\not{I}_{11}}{R^5} + (2-F_a)F_a\Omega_{\rm DR}^2\Big](\delta I_1)_{\rm N} \\ &-\frac{4}{R^6}\big[2(\delta I_{111})_{\rm N} - 3(\delta I_{122})_{\rm N} - 3(\delta I_{133})_{\rm N}\big] \\ &+\frac{1}{c^2}\delta\Omega^2 + O(R^{-7}),\end{aligned} \tag{7.1}$$

where the subscript N denotes the perturbation of Newtonian order. Equation (7.1) is approximated as

$$\Omega^2 = \frac{2M}{R^3}\Big[1 + \frac{3}{5\tilde{R}^2}(2-\alpha_2^2-\alpha_3^2) - \omega_1\tilde{t}_{21} - \omega_2\tilde{t}_{22} + C_{\rm s}\frac{\tilde{R}^3}{2\alpha_2\alpha_3}\delta\tilde{\Omega}^2\Big], \tag{7.2}$$

where $\tilde{t}_{21}$ and $\tilde{t}_{22}$ denote the amplitude of the Lagrangian displacement vectors (3.10) and (3.11), and

$$\begin{aligned}\omega_1 = &\frac{1}{5\tilde{R}}\Big[1 + \frac{3}{5\tilde{R}^2}(2-\alpha_2^2-\alpha_3^2)\Big]\big[1+(2-F_a)F_a\big] \\ &+\frac{6}{35\tilde{R}^3}\Big(3 - \frac{1}{2}\alpha_2^2 + \frac{3}{2}\alpha_3^2\Big),\end{aligned} \tag{7.3}$$

$$\omega_2 = -\frac{12}{35\tilde{R}^3}(\alpha_2^2 - \alpha_3^2). \tag{7.4}$$

From Table II, we can obtain the values in the range of $0.01 \le C_{\rm s} \le 0.05$ as

$$\omega_1\tilde{t}_{21} \sim 0.14\tilde{t}_{21}, \tag{7.5}$$

$$\omega_2\tilde{t}_{22} \sim 0.001\tilde{t}_{22}, \tag{7.6}$$

$$0.0243 \le C_{\rm s}\frac{\tilde{R}^3}{2\alpha_2\alpha_3}\delta\tilde{\Omega}^2 \le 0.121. \tag{7.7}$$

Therefore, if the amplitude $\tilde{t}_{21}$ (and $\tilde{t}_{22}$) is much less than 1, i.e., in the range of $0 \le \tilde{t}_{21} \le 0.1$, the contribution of the octupole deformation to the angular velocity is smaller than that of the 1PN correction. On the other hand, if $\tilde{t}_{21}$

and/or $\tilde{t}_{22}$ have values near or greater than 1, the above perturbative treatments break down. In this case, we cannot estimate the contribution of the octupole moment until we construct the equilibrium figure including the octupole moment.

Moreover, we would like to mention the importance of the velocity potential of 1PN order. If we neglect it, we obtain different features for the ISCO. Accordingly, we must take into account the velocity potential at 1PN order even when we use the ellipsoidal approximation.

From Fig. 5, we can estimate the frequency of gravitational waves at the ISCO as

$$450\ [\mathrm{Hz}] \lesssim \left(\frac{M_*}{1.6M_\odot}\right) f_{\mathrm{GW}} \lesssim 800\ [\mathrm{Hz}], \tag{7.8}$$

if we consider that we have $0 \lesssim n \lesssim 1$ for the polytropic indicies and $0.15 \lesssim C_{\mathrm{s}} \lesssim 0.25$ for the compactness parameter of neutron stars.

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APPENDIX A: EXPLICIT FORM OF δU

The correction of the self-gravity potential induced by the deformation of the binary system is given by [27,19]

$$\delta U = \frac{1}{c^2} \sum_{ij} S_{ij} \delta U^{(ij)}, \tag{A1}$$

where

$$\delta U^{(ij)} = -\sum_k \frac{\partial}{\partial x_k} \int d^3x' \frac{\rho(x')\xi_k^{(ij)}(x')}{|x-x'|}. \tag{A2}$$

Substituting the Lagrangian displacement vectors into Eq. (A2), we separately obtain δU as

$$\delta U = \delta U^{1\to1} + \delta U^{2\to1}, \tag{A3}$$

where

$$\begin{aligned}
\delta U^{1\to1} = \frac{1}{c^2}\Big[& S_{11}(D_{3,3}-D_{1,1}) + S_{12}(D_{3,3}-D_{2,2}) + S_{31}\Big(D_{112,2} - \frac{1}{3}D_{111,1}\Big) \\
& + S_{32}\Big(D_{223,3} - \frac{1}{3}D_{222,2}\Big) + S_{33}\Big(D_{133,1} - \frac{1}{3}D_{333,3}\Big) \\
& - \frac{1}{2}S_0 U^{1\to1}_{,1} + S_{21}\Big(D_{13,3} - \frac{1}{2}D_{11,1}\Big) + S_{22}(D_{13,3} - D_{12,2})\Big],
\end{aligned} \tag{A4}$$

$$\begin{aligned}
\delta U^{2\to1} = \frac{1}{c^2}\Big[& S_{11}(D^{2\to1}_{3,3}-D^{2\to1}_{1,1}) + S_{12}(D^{2\to1}_{3,3}-D^{2\to1}_{2,2}) + S_{31}\Big(D^{2\to1}_{112,2} - \frac{1}{3}D^{2\to1}_{111,1}\Big) \\
& + S_{32}\Big(D^{2\to1}_{223,3} - \frac{1}{3}D^{2\to1}_{222,2}\Big) + S_{33}\Big(D^{2\to1}_{133,1} - \frac{1}{3}D^{2\to1}_{333,3}\Big) \\
& + \frac{1}{2}S_0 U^{2\to1}_{,1} - S_{21}\Big(D^{2\to1}_{13,3} - \frac{1}{2}D^{2\to1}_{11,1}\Big) - S_{22}(D^{2\to1}_{13,3} - D^{2\to1}_{12,2})\Big].
\end{aligned} \tag{A5}$$

Note that the sign of ξ_i for $\delta U^{2\to1}$ is opposite to that of $\delta U^{1\to1}$ if the components of ξ_i are even functions of x_i.

APPENDIX B: PERTURBED TERMS

The definitions of δI_i, δI_{ij}, and δW are

$$\delta I_i \equiv \int d^3x\rho\xi_i, \tag{B1}$$

$$\delta I_{ij} \equiv \int d^3x\rho \sum_l \xi_l \frac{\partial}{\partial x_l}(x_i x_j), \tag{B2}$$

$$\delta W \equiv \sum_{i=1}^{3} \delta W_{ii}, \tag{B3}$$

where

$$\delta W_{ij} = \pi\rho\big(-2B_{ij}\delta I_{ij} + a_i^2 \delta_{ij} \sum_l A_{il}\delta I_{ll}\big). \tag{B4}$$

The components are

$$\delta I_1 = \frac{1}{2c^2}(MS_0 + S_{21}I_{11}), \tag{B5}$$

$$\delta I_{11} = \frac{2}{c^2}\Big(S_{11}I_{11} + \frac{1}{3}S_{31}I_{1111} - S_{33}I_{1133}\Big), \tag{B6}$$

$$\delta I_{22} = \frac{2}{c^2}\Big(S_{12}I_{22} - S_{31}I_{1122} + \frac{1}{3}S_{32}I_{2222}\Big), \tag{B7}$$

$$\delta I_{33} = \frac{2}{c^2}\Big(-S_{11}I_{33} - S_{12}I_{33} - S_{32}I_{2233} + \frac{1}{3}S_{33}I_{3333}\Big), \tag{B8}$$

$$\begin{aligned}\delta W_{11} = \frac{2\pi\rho}{c^2}\Big[& S_{11}\Big\{(a_1^2A_{11} - 2B_{11})I_{11} - a_1^2A_{13}I_{33}\Big\} \\ & +S_{12}(a_1^2A_{12}I_{22} - a_1^2A_{13}I_{33}) \\ & +S_{31}\Big\{\frac{1}{3}(a_1^2A_{11} - 2B_{11})I_{1111} - a_1^2A_{12}I_{1122}\Big\} \\ & +S_{32}\Big(\frac{1}{3}a_1^2A_{12}I_{2222} - a_1^2A_{13}I_{2233}\Big) \\ & +S_{33}\Big\{-(a_1^2A_{11} - 2B_{11})I_{1133} + \frac{1}{3}a_1^2A_{13}I_{3333}\Big\}\Big],\end{aligned} \tag{B9}$$

$$\begin{aligned}\delta W_{22} = \frac{2\pi\rho}{c^2}\Big[& S_{11}(a_2^2A_{12}I_{11} - a_2^2A_{23}I_{33}) \\ & +S_{12}\Big\{(a_2^2A_{22} - 2B_{22})I_{22} - a_2^2A_{23}I_{33}\Big\} \\ & +S_{31}\Big\{\frac{1}{3}a_2^2A_{12}I_{1111} - (a_2^2A_{22} - 2B_{22})I_{1122}\Big\} \\ & +S_{32}\Big\{\frac{1}{3}(a_2^2A_{22} - 2B_{22})I_{2222} - a_2^2A_{23}I_{2233}\Big\} \\ & +S_{33}\Big(-a_2^2A_{12}I_{1133} + \frac{1}{3}a_2^2A_{23}I_{3333}\Big)\Big],\end{aligned} \tag{B10}$$

$$\begin{aligned}\delta W_{33} = \frac{2\pi\rho}{c^2}\Big[& S_{11}\Big\{a_3^2A_{13}I_{11} - (a_3^2A_{33} - 2B_{33})I_{33}\Big\} \\ & +S_{12}\Big\{a_3^2A_{23}I_{22} - (a_3^2A_{33} - 2B_{33})I_{33}\Big\} \\ & +S_{31}\Big(\frac{1}{3}a_3^2A_{13}I_{1111} - a_3^2A_{23}I_{1122}\Big) \\ & +S_{32}\Big\{\frac{1}{3}a_3^2A_{23}I_{2222} - (a_3^2A_{33} - 2B_{33})I_{2233}\Big\} \\ & +S_{33}\Big\{-a_3^2A_{13}I_{1133} + \frac{1}{3}(a_3^2A_{33} - 2B_{33})I_{3333}\Big\}\Big].\end{aligned} \tag{B11}$$

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